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ON THE DETERMINATION OF THE MAGNETIC BEHAVIOR OF THE FINELY DIVIDED CORE OF AN ELECTROMAGNET WHILE A STEADY CURRENT IS BEING ESTABLISHED IN THE EXCITING COIL.

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More than fifty years ago Helmholtz established, on theoretical grounds, the now familiar equations for the manner of growth of a current in a circuit of constant inductance under a given electromotive force, and proved by a brilliant series of experiments [1] that the predictions of this theory were fulfilled in practice. It appeared, in particular, that if a circuit of resistance r containing a constant electromotive force, E, were closed at the origin of time, the current, I, would be given by the expression

$$\frac{E}{r}(1 - e^{-\frac{rt}{L}}), \tag{1}$$

if L were the "potential of the circuit upon itself," that is, the self-inductance. The "induced current" (i) would satisfy the equation

$$i = \frac{L}{r} \cdot \frac{dI}{dt} = \frac{E}{r} \cdot e^{-\frac{rt}{L}}. \tag{2}$$

If, therefore, I were plotted against the time, the resulting curve ($OGQKC$, Figure 1) would have as asymptote the straight line (ZC) parallel to the t axis at a distance E/r above it; the current in the circuit at any time (OP) would be given by the corresponding

[1] F. E. Neumann, Abh. d. Berl. Akad. 1845 and 1847; Helmholtz, Die Erhaltung der Kraft, 1847; Pogg. Ann., **83**, 1851; **91**, 1854; Phil. Mag., **42**, 1871.

ordinate (PQ) of the curve and the instantaneous value of the induced current by the distance (NQ) at that time, of the curve from the asymptote. The whole "amount" of the induced current up to the given time would be represented by the shaded area (A) shut in by the curve, the asymptote, and the ordinates, $t = 0$, $t = OP$. If the electromotive force were suddenly shunted out of the circuit after the current had reached its final value, the "extra current" would have the value

$$\frac{E}{r} \cdot e^{-\frac{rt}{L}}. \tag{3}$$

Helmholtz also studied the "forms" of the currents induced in the secondary circuit of a small induction coil at the making and breaking of the primary circuit, and, by using in the apparatus iron cores, some of which were solid and some finely divided, he showed that the effect of eddy currents in the iron upon the apparent duration of the induced currents might be very appreciable. The results of Helmholtz's experiments were confirmed with the aid of other apparatus, during the next thirty years,[2] by a number of physicists.

The mathematical treatment of the subject begun by Neumann and Helmholtz was in 1854 pushed somewhat farther by Koosen, and in 1862 E. du Bois-Reymond[3] published an elaborate discussion of the equations laid down by Helmholtz for the determination of the currents in two neighboring circuits of constant self-inductances (L_1, L_2) and constant mutual inductance (M), and gave the solutions of the simultaneous equations

$$\left.\begin{aligned} L_1 \cdot \frac{dI_1}{dt} + M \cdot \frac{dI_2}{dt} + r_1 I_1 = E_1, \\ M \cdot \frac{dI_1}{dt} + L_2 \cdot \frac{dI_2}{dt} + r_2 I_2 = E_2, \end{aligned}\right. \tag{4}$$

corresponding to a number of different sets of physical conditions, in nearly the forms in which they now appear in textbooks. Du

[2] Felici, Ann. de Chimie, **34**, 1852; N. Cimento, **3**, 1856; **9**, 1859; **12**, 1874; **13**, 1875. Cazin, Compt. Rend., **60**, 1865; Ann. de Chimie, **17**, 1869. Guillemin, Compt. Rend., **50**, 1860. Bertin, Mem. de la Soc. des Sc. Nat. Strasbourg, **6**, 1865. Bazzi and Corbianchi, N. Cimento, **4**, 1878. Bartolli, Mem. d. Acc. d. Lincei, **6**, 1882. Bazzi, Att. d. Acc. d. Lincei, **6**, 1882. Lemström, Pogg. Ann., **147**, 1872. V. Ettingshausen, Pogg. Ann., **159**, 1876.

[3] Koosen, Pogg. Ann., **91**, 1854. E. du Bois-Reymond, Monatsberichte d. Berl. Akad., 1861, 1862. Brillouin, Thèse, 1880; Jour. de Phys., **10**, 1881; Compt. Rend., 1882.

Bois-Reymond showed that if the secondary circuit contained no battery, and if, after the primary current had been fully established, its circuit were *suddenly* broken, the current induced in the secondary circuit would have a form like that of the dotted curve (P) in Figure 2; if after a few seconds the primary circuit were again closed, the secondary current when plotted against the time would yield a curve either like Q or like S in the same diagram. The lines in this familiar figure have been drawn to scale for a certain pair of circuits the self-inductances of which are equal, fixed quantities and the resistances also fixed. Q, R, S correspond to three different values of the mutual inductance (M), which are respectively half as great, nine tenths as great, and equal to the self-inductance (L)

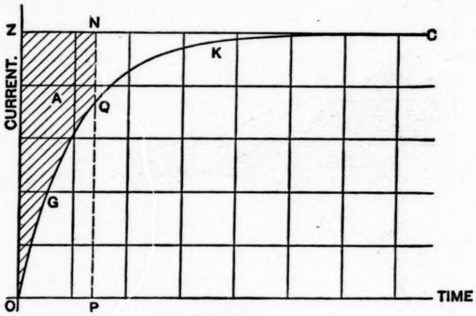

FIGURE 1.

If the current is expressed in absolute units (absamperes) and the time in seconds, the shaded area represents the change in the total flux of magnetic induction through the circuit, during the time OP.

of either circuit. These curves show the currents induced in the secondary circuit when the primary is made; the crest of any such curve comes earlier the larger the value of M. The curve P, which represents a current induced in the secondary circuit when the primary circuit is broken, is drawn for the case $M = \frac{1}{2}L$, and therefore corresponds to the curve Q; E. du Bois-Reymond called attention to the fact that in such problems as this the areas V and W must be equal. The curves like P corresponding to R and S could be found merely by exaggerating all the ordinates of P in the ratio 9/5 or the ratio 2.

From the early days of induction coils, iron cores had been used to increase the mutual inductance of the circuits, and, soon after Helmholtz had given the equations for the currents in neighboring

circuits of constant inductances, coils containing iron were studied from the point of view of the principles which he had laid down. Helmholtz's own experiments and those of others soon showed, however, that the introduction of masses of magnetic metal into the space within the coils complicated very much their action. It ap-

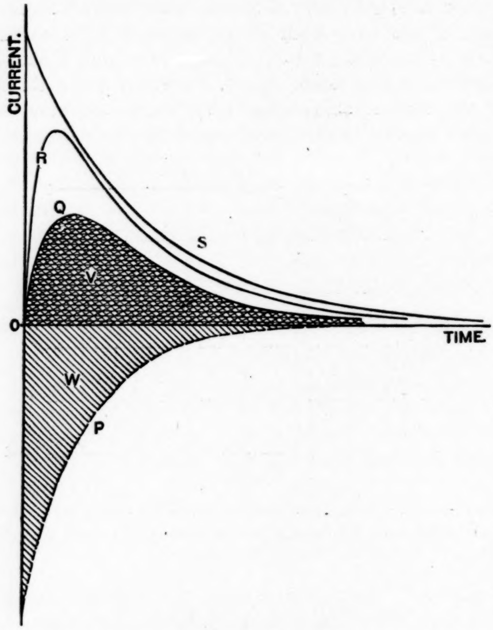

Figure 2.

The curves Q, R, S represent for different relative values of the mutual inductance the current induced in the secondary circuit of a certain induction coil without iron, when the primary circuit is suddenly closed.

peared that the existence of eddy currents in the iron, if the coil were solid, and the fact that the counter electromotive force in a circuit — as measured by the time rate of change of the flux of magnetic induction through it — is by no means proportional to the rate of change of the intensity of the current if a circuit "contains iron," made the simple theory of Helmholtz inapplicable, as he himself had foreseen

that it would be. The subject interested many investigators,[4] who found it easy to exhibit the disturbing effects of eddy currents in hindering rapid magnetic changes in solid masses of iron and in thus modifying the characters of the induced currents; but it was not until much work had been done by many persons on the phenomena attending magnetic induction in iron that the theory of the alternate current transformer which had meanwhile come to be of much practical importance was very well understood. With the general introduction of dynamo-electric machinery the magnetic behavior of the different kinds of iron used in its manufacture became of practical interest, and several different magnetometric and ballistic methods of studying permeability were invented and employed in making the necessary measurements upon relatively small pieces of the metal.

Soon after the first hysteresis diagrams had been obtained as a result of experiments either on comparatively thin iron or steel rings, or on long, fine wires, it was found by engineers that, on account of the considerable time required to establish a steady current in the coil of a large electromagnet to which a given electromotive force had been applied, the "reversed current," and even the "step-by-step" ballistic methods which had proved effective in the cases of slender toroids, were, in their old forms at least, not well fitted for studying the magnetic properties of such massive closed iron circuits as frequently occurred in practice. When there was a gap in such a circuit, the problem, of course, offered no difficulty,

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[4] Faraday, Researches, 1831, 1832, 1846. Lenz, Pogg. Ann., **31**, 1834. Henry, American Journal of Science, 1832; Phil. Mag., **16**, 1840. Dove, Pogg. Ann., **43**, 1838; **54**, 1841; **56**, 1842. Beetz, Pogg. Ann., **102**, 1857; **105**, 1858. Plücker, Pogg. Ann., **87**, 52; **94**, 1855. Rayleigh, Phil. Mag., **38**, 1869; **39**, 1870; **23**, 1887; **22**, 1886. Bichat, Ann. de l'École Norm., **10**, 1873. Sinsteden, Pogg. Ann., **92**, 1854. Magnus, Pogg. Ann., **38**, 1836; **48**, 1839. Schneebeli, Bull. de la Soc. des Sc. Nat. de Neufchatel, **11**, 1877. Blaserna, Giornn. di Sc. Nat., **6**, 1870. Maxwell, Electricity and Magnetism, **2**, iv. Donati and Poloni, N. Cimento, **13**, 1875. Stoletow, Phil. Mag., **45**, 1873. Auerbach, Wied. Ann., **5**, 1878. Rowland, Phil. Mag., **46**, 1873; **48**, 1874. Thomson, Phil. Trans., **165**, 1875. J. Hopkinson, Phil. Trans., **176**, 1885. Von Waltenhofen, Pogg. Ann., **120**, 1863. Warburg, Wied. Ann., **13**, 1881. Wiedemann, Lehre von der Elektricität. Ewing, Phil. Trans., **176**, 1885; Proc. Roy. Soc., 1882, Magnetic Induction in Iron and other Metals. Du Bois, The Magnetic Circuit. Fleming, The Alternate Current Transformer. Ewing and Low, Proc. Royal Soc., **42**, 1887; Phil. Trans., **180**, 1889. Du Bois, Phil. Mag., 1890. Oberbeck, Wied. Ann., **22**, 1884. J. and E. Hopkinson, Phil. Trans., **177**, 1886. Jouaust, Compt. Rend., **139**, 1904. E. Hopkinson, Brit. Assoc. Report, 1887. Tanakadaté, Phil. Mag., 1889. Wilson, Proc. Royal Soc., **62**, 1898. Baily, Phil. Trans., **187**, 1896. Many other references may be found in these sources.

but when large iron frames were completely closed, it became the custom, in satisfying commercial contracts, to attempt to get information about the permeability of the metal as a whole from tests, under given conditions, upon small, thin specimen pieces made as nearly as possible of the same material as the original, or else cut from it. It was usually impossible, however, to be sure that the temper of the small piece was sufficiently like that of the mass to make it a fair representative of the whole, and the preparation of the specimens was often troublesome, so that some more practical method of procedure was seen to be desirable,[5] and it seems to have occurred to a number of different persons independently that a good deal might be learned about the magnetic properties of the core of an electromagnet if one determined the manner of growth of a current in an exciting coil of a given number of turns wound closely about the core, when, under given initial conditions, a constant, known, electromotive force was applied to the coil circuit.

The Determination of some of the Magnetic Properties of the Core of an Electromagnet from the March of a Current in the Exciting Coil.

If, at any instant, the total flux of magnetic induction through the n turns of the exciting coil of an electromagnet is N (maxwells), if r is the resistance of the coil circuit (in ohms), i the current in it (in amperes), and E the applied electromotive force (in volts), then

$$E - \frac{1}{10^8} \cdot \frac{dN}{dt} = ri, \tag{5}$$

or

$$\frac{dN}{dt} = 10^8 \cdot r \left(\frac{E}{r} - i \right); \tag{6}$$

and if the final value (E/r) of the current be denoted by i_∞ and the change in N during the time interval t_1 to t_2 by $N_{1,2}$,

$$N_{1,2} = r \cdot 10^8 \cdot \int_{t_1}^{t_2} (i_\infty - i) \, dt. \tag{7}$$

If, now, i be plotted against the time in a curve s (Figure 3) in which l centimeters parallel to the axis of abscissas represent one second, and an ordinate m centimeters long one ampere, the curve

[5] Drysdale, Jour. Inst. Elec. Engineers, **31**, 1901.

will have an asymptote, CY, parallel to the axis of abscissas, at a distance, KC, from it corresponding to E/r amperes, and, if OK represents the time t_1, and OL the time t_2, the area $FGDC$, or $A_{1,2}$, expressed in square centimeters, is equal to

$$lm \int_{t_1}^{t_2} (i_\infty - i)\, dt, \tag{8}$$

so that
$$N_{1,2} = \frac{r \cdot 10^8 \cdot A_{1,2}}{lm} = \frac{10^8 \cdot E \cdot A_{1,2}}{lm \cdot i_\infty}. \tag{9}$$

In practice N usually differs from $n\phi$, where ϕ is the induction flux through the iron core of the electromagnet alone, by only a small fraction of itself, and, if a is the area of the cross section of the core at any point, a certain average value of B, the induction, can be obtained from the expression N/na, though in such cores as are used in large transformers, H, and therefore B, would probably have very different values at different points of the section. Really N is greater than $n\phi$ by the amount of the magnetic flux, in the air about the core, through the turns of the exciting coil, caused by the current in the coil itself or by neighboring currents, if there are such.

Using this theory, a good many persons have studied at various times the magnetic properties of different large masses of iron, and in 1893 Professor Thomas Gray of Terre Haute published in the Philosophical Transactions of the Royal Society a long series of very beautiful current curves,[6] obtained, with simple apparatus handled with great skill, from a 40 K. W. transformer belonging to the Rose Polytechnic Institute. A number of diagrams[7] showing the manner of growth of currents in the exciting coils of large electromagnets with solid cores have been printed within the last dozen years; of these the curves given by Dr. W. M. Thornton are especially interesting.

If to the coil of an electromagnet, in series with a rheostat of resistance r, a given electromotive force be applied, and if r be then reduced by steps, at intervals so long that one is sure that the final current belonging to each stage has been practically attained, the curve which has elapsed times for abscissas and the corresponding

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[6] T. Gray, Phil. Trans., **184**, 1893.

[7] Hopkinson and Wilson, Journal of the Institute of Electrical Engineers, **24**, 1895. Thornton, Electrical Engineer, **29**, 1902; Phil. Mag., **8**, 1904; Electrician, 1903 Peirce, These Proceedings, **41**, 1906. Several figures from this last paper are here reproduced.

values of the strength of the current for ordinates, will have the general form of the line *U* in Figure 4, though, if the core be so large that the building up time at each stage is long, the diagram will be much drawn out horizontally. The curve which shows the march of the current when the electromotive force is applied directly to the coil without the intervention of the rheostat will resemble line *V* in the same figure. The exact forms of these curves depend, of course, upon

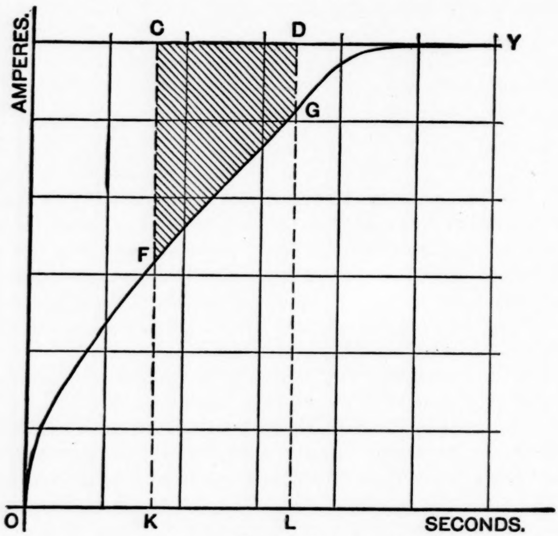

FIGURE 3.

If l centimeters parallel to the horizontal axis represent one second, and an ordinate m centimeters long one ampere, $A \cdot 10^8 \cdot r/lm$ (where A is the area, in square centimeters, of $CDGF$) represents the change in the magnetic flux through the circuit during the interval KL.

the magnetic state of the core at the outset, and will be very different if the iron has been thoroughly demagnetized before the observation is made, or if it be strongly magnetized. Figure 5, which illustrates this fact for some *V* curves, records some measurements made upon a 15 K.W. transformer (*R*) belonging to the Lawrence Scientific School. In the case represented by each line the core was previously magnetized in one direction with the full strength of the current, and the circuit was then broken and left open for a few seconds. With the

electromotive force in it unchanged in intensity, but in some instances changed in direction, the circuit was then closed again and a current curve obtained. If the electromotive force has its old direction, such a curve is said to be "direct"; if the new direction is the opposite of the old, the curve is called "reverse." In one case the magnetic journey of the core during the rise of the current is represented approximately by the portion *PFM* of the corresponding hysteresis diagram (Figure 6); in the other case the journey follows the arc *QUZM*. Lines 1, 2, and 4 in Figure 5 are reverse lines, while 3 and 5 are direct.

In Figure 4 the line OY corresponds to the final value (i_∞) of the current, and if its length in centimeters is $m\ i_\infty$ and if A is the area in square centimeters shut in by OY, YX, and V, it is evident that in the

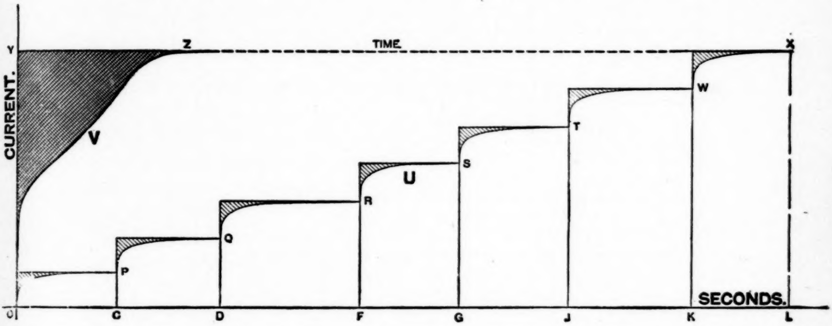

FIGURE 4.

Curves which represent the growth of the current in the exciting coil of an electromagnet when (V), the circuit which has the resistance r, is closed and left to itself; and when (U), the circuit, is closed when it has a comparatively large resistance, which is then reduced to r by steps.

case represented by V the whole change in induction flux through the turns of the coil due to the current is

$$\frac{10^8 \cdot E \cdot A}{l \cdot OY}.$$

In the case represented by the line U, (10^8. E/l) times the sum of the terms formed by dividing each of the small shaded areas by the ordinate, expressed in centimeters, of its upper straight boundary, gives the change in the induction flux through the turns of the coil due to the current when it grows in the manner indicated. Of course if the

current is not allowed time to attain its final value at each stage, a serious error may be introduced.

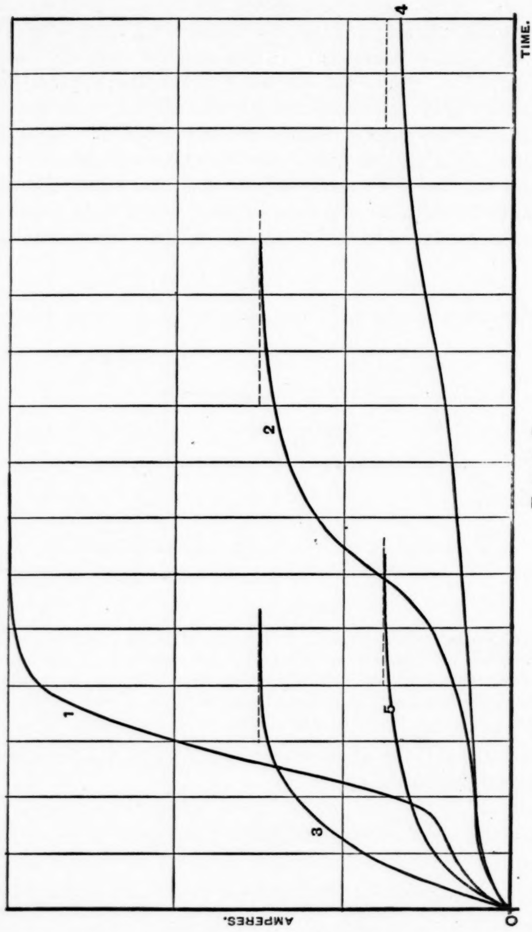

FIGURE 5.

Oscillograph records of direct and reverse curves for the magnet *R* taken with an exciting coil of 340 turns: the resistance of the circuit was kept constant, and the applied electromotive force was adjusted so as to make the final values of the current 3.00 amperes, 1.50 amperes, and 0.75 amperes.

The amount of flux which, in a given large mass of iron, in a given magnetic condition at the outset, corresponds to a current of given final

strength in the exciting coil, usually depends in some slight degree upon the manner of growth of the current. If after a large core has been magnetized in one direction by the steady application of a given electromotive force until the current has reached its full value, the exciting circuit be broken, and, after the direction of the electromotive force has been reversed, closed again, it sometimes happens that the magnetic flux after the new current has attained its maximum value is slightly less when the current follows the course of the curve V than when it grows by short stages in the manner indicated by the curve U. If, however, there are but two or three steps, the difference is, as a rule, of no practical importance, and if one has a suitable oscillograph or other recording instrument, it is possible to get a set of current curves for any given maximum value of the current from which an extremely good statical hysteresis diagram may be obtained for the core.

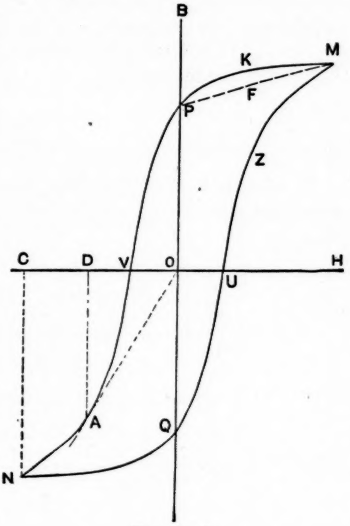

FIGURE 6.

When a direct current curve is taken, the core of the electromagnet makes a magnetic journey represented approximately by the arc PFM; in the case of a reverse curve the core follows the line $QUZM$.

If while a steady current from a constant storage battery of voltage E is passing through the coil of an electromagnet, the resistance of the coil circuit be suddenly increased to a new value r_1, so that the current (i) will ultimately fall to a lower value represented by ON in Figure 7, the current curve, which has been a horizontal line, sinks in such a manner as to become asymptotic to the horizontal line NB. At any instant after the change,

$$E - \frac{dN}{dt} = r_1 i, \tag{10}$$

in absolute units, so that in volts, ohms, amperes, and maxwells,

$$N_{0,1} = 10^8 \int_{t_0}^{t_1} (E - r_1 i)\, dt = 10^8 \cdot r_1 \cdot \int_{t_0}^{t_1} (i_1 - i)\, dt. \tag{11}$$

If an abscissa l centimeters long corresponds to one second, and an ordinate m centimeters represents one ampere, and if $A_{0,1}$ stands for the area in square centimeters bounded by the current curve, the asymptote, and ordinates corresponding to the times t_0, t_1, the change in the flux of magnetic induction through the circuit during this time-interval is (in maxwells)

$$\frac{10^8 \cdot r_1 \cdot A_{0,1}}{lm}. \tag{12}$$

If, after a current has been built up by stages in the coil of an electromagnet, in the manner indicated by curve U of Figure 4, the

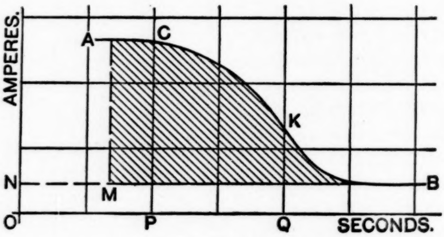

Figure 7.

The shaded area represents on a certain scale the change in the flux of magnetic induction through a circuit when the resistance of the circuit is suddenly increased and then kept constant.

process be reversed, and the resistance of the circuit be increased by steps, the current curve will look very much as the curve U would if looked at from the wrong side of the paper when upside down.

As has already been stated, it is possible to get slightly different hysteresis diagrams for a massive core originally demagnetized, when the current is made to change from a given positive limit to the negative limit in different ways; and it is important, in predicting the behavior of a magnet which is to be used for a given purpose, to employ in computation the hysteresis diagram which corresponds to the particular magnetic journey which the core will take in practice. A single carefully made curve of the U type with a dozen steps will, however, give a result good enough for any commercial purpose, though

my own experience shows that it is not always easy to measure all the small areas, especially the lower ones, with the desirable accuracy, when the width (OY) of the whole diagram is only 12 or 14 centimeters.

If in the U diagram there is only one intermediate stage, and if the core is in a given magnetic condition at the outset, the change in the magnetic flux, due to a current of given final value, ought not to differ by more than perhaps a fraction of one per cent from the corresponding change when there is no intermediate step and the case is represented by V. Sometimes a series of U diagrams, each with but one

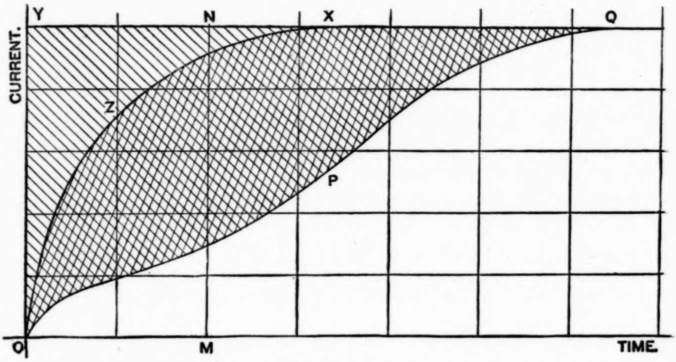

FIGURE 8.

The areas between the asymptote and the curves Z and P are proportional to the changes of magnetic flux through the circuit caused by direct and reverse currents of the same final strength.

intermediate step, at a place determined by a proper choice of r, may be made to yield very accurate information about the permeability of the large mass of metal which will suit some special use of the magnet.

Figure 8, which resembles in general design some diagrams given by Dr. Thornton, shows a "direct curve" (Z) and a "reverse curve" (P) for a certain magnet. The area $OZXY$ represents the change of magnetic induction when the core covers the arc PFM (Figure 6) on the hysteresis diagram belonging to the journey; the area $OPQXY$ represents the change of magnetic flux when the core takes the journey corresponding to the arc $QUZM$ on the hysteresis diagram. The doubly shaded area represents the flux change corresponding to the line $QUZMKP$.

The Uses of Exploring Coils wound upon the Core of an Electromagnet.

If an electromagnet, in addition to its exciting coil, has another wound about its core, and if the observer has means of obtaining the intensity (i') of the current induced in this secondary coil, for given current changes in the exciting coil, as a function of the time, it is easy to study the magnetic properties of the core under the circumstances of the experiment. Let there be n' turns in the secondary coil, let the resistance of its circuit be r' ohms, and let N' be the total induction flux, in maxwells, through the turns of the coil at the time t, then if i' is measured in amperes

$$\frac{dN'}{dt} = -\, 10^8 \cdot r' \cdot i'. \tag{13}$$

If i' be plotted against the time in a curve in which l' centimeters parallel to the axis of abscissas represent one second and an ordinate m' centimeters long one ampere, and if $A'_{1,2}$ represents the area between the curve, the axis of abscissas and the ordinates corresponding to the time t_1, and t_2, we have in absolute value,

$$N_2' - N_1' = 10^8 \cdot r' \int_{t_1}^{t_2} i' \cdot dt = \frac{10^8 \cdot r' \cdot A'_{1,2}}{l'm'} = q' \cdot A'_{1,2}, \tag{14}$$

where q' is a known constant.

When the primary current (i) in the exciting coil is growing, the current in the secondary coil has a direction opposite to that of i, and it is often desirable to emphasize this fact in a diagram by drawing the i, t and i', t curves on opposite sides of the axis of abscissas; but if the relative values of i and i' are alone to be considered, it is sometimes more convenient to disregard their relative directions. If in any case the current in the exciting coil of an electromagnet be made to grow in the manner indicated by curve U in Figure 4, the i', t diagram will consist (Figure 9) of a set of detached areas on the t axis. The sum of any number of these areas when multiplied by $10^8\, r'/l'\, m'\, n'$ gives approximately the whole change in the induction flux through the core up to the corresponding time, from the outset. In the "step-by-step" ballistic method of determining the permeability of a closed ring of rather small cross section the areas represented by the shaded portions of Figure 9 are determined by discharging the induced current through a calibrated ballistic galvanometer of long period, and assuming that the first elongations of the suspended system measure

these areas directly. As will appear in the sequel, it is possible, though not very easy, to get good results in this way, even if the cross-section of the laminated core is as great as, say, 800 square centimeters; for this, however, a properly constructed galvanometer is required.

The "time constant" of a circuit in which a current of given final intensity is to be established is shorter the higher the electromotive force used to generate the current; it is desirable, therefore, to employ a battery of rather high voltage and to reduce the current by non-inductively wound resistance in series with the exciting coil of the electromagnet. If a moving coil galvanometer is used, it is often necessary to correct for the effect of the counter electromotive force induced in the coil as it swings in the field of its own permanent magnet, and

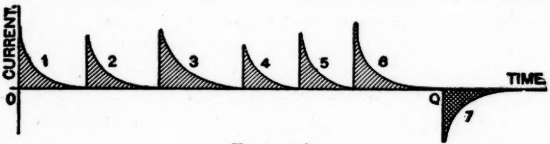

FIGURE 9.

A portion of the record of an oscillograph in the circuit of a secondary coil wound on the core of an electromagnet when the current in the exciting coil is made to change by sudden steps in the determination of a hysteresis cycle.

it is always necessary to use steps so short and to make the period of the galvanometer so long (perhaps 300 or 500 seconds) that the practical duration of the induced current may be small in comparison. It is usual to send the current to the exciting coil by means of a commutator and a long series of manganine resistance coils capable of carrying the desired currents; these coils are often mounted in a frame furnished with some device by which any or all of them can be shunted out of the circuit at pleasure. Two rheostats, made for this purpose some years ago by the Simplex Electric Company, have been found by the staff of the Jefferson Physical Laboratory very satisfactory in practice. By means of such a set of coils as those just described, one may easily get either a progressive, step-by-step increase or decrease in the current, or a reiteration of any particular step. One convenient way of arranging the apparatus for the repetition at pleasure of any desired step has been recently described by A. H. Taylor.[8] The method of rever-

[8] A. Hoyt Taylor, Phys. Rev., **23**, 1906. Mordey and Hansard, Elect. Engineer, **34**, 1904. Searle and Bedford, Phil. Trans., **198**, 1902. Drysdale, Jour. Inst. Elect. Engineers, **31**, 1901. Lamb and Walker, Electrical Review, **48**, 1901.

sals is usually unsatisfactory with large cores. A set of adjustable electrolytic resistances fitted for carrying heavy currents is often useful.

In the case of a very large closed electromagnet, even if the core be laminated, it is extremely difficult to get very useful results by aid of a ballistic galvanometer of short period, but if one has a suitable oscillograph or other recording instrument at hand, it is easy to obtain a diagram something like that shown in part in Figure 9, though it is necessary to make sure that the intervals between the steps, unlike those in this figure, are long enough to record the whole of each induced current.

If the primary current (i, t) curves are to be used in studying the magnetic changes in the core of an electromagnet, the sensitiveness of the oscillograph must be so adjusted that the deflection due to the largest value of the current (U, Figure 4) will make a record on the paper; if the (i', t) curves are to be used, the steps may be as numerous as one likes, and the sensitiveness of the recording instrument may be so great that, starting from the base line, the record of the highest induced current shall just fall on the drum. In this latter case the areas to be measured may be made so large that any uncertainty as to the exact time when any induced current may be considered to end is unimportant. When many records are taken on the same paper, the drum has an opportunity to revolve a good many times during the operation, and it is not always easy to decipher the complicated maze of curves. Of course the fact that an electromagnet has a closed secondary circuit modifies somewhat the form of the building-up curve in the primary, but, theoretically at least, this should not affect the value of the magnetic flux due to the primary current if its final intensity is given, and the difference is inappreciable if there are only a few turns in the secondary coil.

Instead of changing the resistance in the primary circuit suddenly, at each step, Dr. Thornton, in dealing with the frames of some very large dynamos, made each step gradually, by moving an electrode slowly in a trough of acidulated water from one stopping place to another. Figure 10 is a close copy of one of his records published in the "Philosophical Magazine" for 1904.

Fluxmeters and Quantometers.

Given an amperemeter of the ordinary d'Arsonval type, in which an open-frame, low resistance, unshunted coil swings in the strong magnetic field between an interior soft iron core and the hollowed-out jaws of a powerful magnet, it is often possible to make the controlling

springs so weak that if the coil circuit be suddenly closed on itself while the coil is in motion, the damping effects of the induced currents will bring the coil almost instantly to rest wherever it may happen to be, and, until the circuit is broken, the coil will keep its position fairly well. Several years ago Dr. R. Beattie [9] showed that if the ends of a low resistance exploring coil (A) be electrically connected with an instrument of this kind, and if the flux of magnetic induction through A be changed during the time interval T by an amount N, the coil will move from its initial position to a new position through an angle proportional to N and, apart from pivot friction, practically independent, within wide limits, of T.

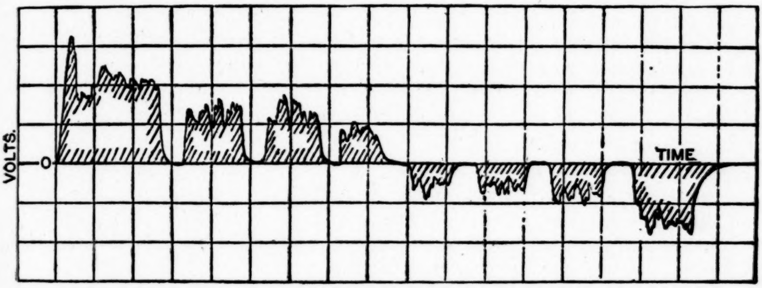

FIGURE 10.

Typical record for half a hysteresis loop, given by Dr. Thornton.

The "quantometer" first made by Dr. Beattie had a coil of twenty-four and a half turns wound on a metal frame and suspended on a single needle point between the poles of an electromagnet; the ends of the coil dipped into mercury cups fixed to the case of the instrument. In the kind of fluxmeter now common, the coil is hung by a silk fibre (or a quartz thread) from a spring, so as to avoid pivot friction; a permanent magnet is used, and the current is led into and out of the coil through helices of very fine silver or copper gimp; the resistance of this gimp is sometimes much greater than that of the coil itself, and for laboratory use it is often well to employ mercury cups, as Dr. Beattie did, so arranged as to minimize the disturbing effects of capillarity. The original quantometer had a resistance of only one ohm.

Many persons who have attempted to use very strong electromagnetic fields in d'Arsonval galvanometers have found that it is very difficult

[9] R. Beattie, Electrician, Dec., 1902.

to procure insulated copper or silver wire for the suspended coil so free from paramagnetic properties that the coil shall not have a permanent "set" in the field, too strong to be conveniently controlled by the torsion of the gimp through which the current enters the coil. In the case of a quantometer where there is practically no controlling moment from the suspending fibre, the paramagnetic properties of the coil may be very troublesome; and in some of the most recent instruments the angular movements of the coils, due to given changes of induction through the turns of the exploring coils, are somewhat different according as the movement is towards the left or towards the right. If a telescope and scale be set up in such a position that the behavior of the coil can be watched after it has moved through a considerable angle, urged by a sudden, definite change of flux in the exploring coil, it will often be found that the coil does not remain even approximately at rest, but moves steadily and so rapidly that a considerable error is introduced if the given change of flux through the exploring coil is made slowly. It is desirable, therefore, to test an instrument of this kind carefully before using it.

If great accuracy is not required, a good fluxmeter, of some standard make, and of sensitiveness suited to the work to be done, is, in experienced hands, a most useful instrument; the time needed to establish a current of given strength in the coil of a large electromagnet with a solid core may be several minutes, but a very good fluxmeter will, nevertheless, show directly, with an error of not more than 2 per cent, the change of magnetic flux through the core.

If the fluxmeter coil is not wound on a closed metal frame, the mutual damping effect of currents in the coil and in the core which it surrounds are not always effective unless the resistance of the external circuit, made up of the exploring coil and its leads, is fairly small compared with the resistance of the suspended coil itself. An instrument, therefore, which works very well with an exploring coil of a small number of turns often becomes quite useless when, in order to get the required sensitiveness, the observer tries to employ an exploring coil made of many turns of fine wire. On the other hand, if a fluxmeter of this kind is too sensitive for a given piece of work, it is not always easy to reduce the sensitiveness quickly.

If the flux changes to be measured are large, it is often convenient to have a fluxmeter the coil of which consists of a few turns either wound on a copper frame or else accompanied by several turns of stout wire closed on themselves. It is possible to use such an instrument with many different exploring coils and to change its sensitiveness within wide limits by varying the resistance of the external circuit.

In doing a small part of the work described below, I was able to use either a Grassot Portable Fluxmeter, or a certain fixed laboratory fluxmeter (F) furnished with a tall chimney to hold the 140 centimeter long fibre by which the coil was suspended. The cast-iron magnet of this last mentioned instrument had, when finished, the form shown in plan in Figure 11 and was 45 mms. thick. The casting was made with a web connecting the poles, and this was removed after the hole for the coil had been cut out and finally reamed to a diameter of exactly 5 cms. on a Browne and Sharpe milling machine. The magnet was hardened and treated by Mr. G. W. Thompson, the mechanician of the Jefferson Physical Laboratory, who has had much experience in this kind of work. During the process the poles were held in position by an iron yoke. The core (shaded in the diagram) within the coil is 41.3 mms. in outer diameter, and is about 7 mms. thick. The instrument was constructed and set up by Mr. John Coulson, who has helped me in countless ways during the progress of the work. It was comparatively easy to substitute one of the set of coils belonging to this fluxmeter for another. For certain purposes it was convenient to have a coil of 200 turns of stout insulated wire which was wound about the magnet, though the latter had a large permanent moment.

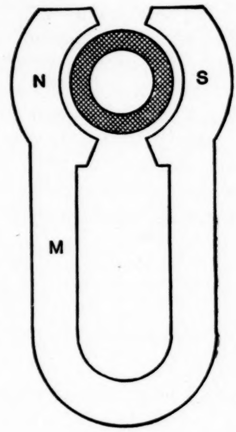

FIGURE 11.

Plan of one of the permanent magnets of the fluxmeter F; the shaded area represents the cross-section of the soft iron core.

THE COEFFICIENTS OF SELF-INDUCTION OF A CIRCUIT WHICH HAS AN IRON CORE.

When many years ago it was found that the induction B at a given point in a piece of iron exposed to a given magnetic field H is not only not in general proportional to the intensity of the exciting force, but is not even determined when H is given, it became evident that no such constant can exist in the case of an inductive circuit which "contains" a magnetic metal as was assumed in the conception of Neumann's

"Electrodynamisches Potential,"[10] and that the different common definitions of self-induction, when applied to an electromagnet of the usual form, really describe physical quantities which are widely different from one another. The ambiguity in the use of the term "self-induction" still exists, and it will be convenient in this paper to adopt the notation used by Sumpner[11] in his article on "The Variations of the Coefficients of Induction." If, in absolute value, I is the strength of a current growing in the coil of an electromagnet with laminated core, if N is the total flux of magnetic induction through the turns of the coil, and e the counter electromotive force of induction, we may call the ratio of e to the time rate of change of the current, L_1, the ratio of N to the current, L_2, and the ratio, to I^2, of twice the contribution (T) made by the current to the energy when there are no other currents in the neighborhood, L_3, so that

$$\begin{aligned} e &= L_1 \cdot \frac{dI}{dt}, \quad N = L_2 \cdot I, \quad L_1 = \frac{dN}{dI} \\ e &= \frac{d(L_2 \cdot I)}{dt}, \quad T = \tfrac{1}{2} L_3 \cdot I^2, \quad L_1 = L_2 + I \cdot \frac{dL_2}{dI}. \end{aligned} \tag{15}$$

If then for a particular magnetic journey, taken at a given speed, N is given as a function of I in the form of a curve like OPQ in Figure 12, the value, at any point P on the curve, of L_1 is the slope of the curve or the tangent of the angle XKP; the value of L_2 at P is the slope of the line OP or the tangent of the angle XOP; the value of L_3 is the ratio of twice the curvilinear area OPD to the area of the square erected on OJ. Similar definitions are sometimes given for such a magnetic journey as is represented by the line $MGPQ$ of Figure 13.

In the paper just cited Sumpner gives a very interesting graphical method of constructing a curve which shall show the manner of growth of the current in the coil of the electromagnet when the curve which connects N and I is given.

The Electromagnets used in doing the Work described below.

A number of electromagnets were used in carrying on the experimental work described in this paper.

Though the investigation had to do primarily with magnets the cores of which were laminated or otherwise finely divided so as to get

[10] Neumann, Abh. d. Berl. Akad., 1845.
[11] Sumpner, Phil. Mag., **25**, 1888.

rid in great measure of the disturbing effects of eddy currents, one or two large magnets with massive cores were useful for purposes of comparison. One of these (*P*), which weighs about 1500 kilograms, has the general shape shown in Figure 14. The outside dimensions of the frame proper are about 101 cms. × 80 cms. × 40 cms. The base is of cast iron and of rectangular cross-section (20 cms. × 40 cms.), the cylindrical arms are of soft steel 25 cms. in diameter, the rectangular pole pieces are 4.5 cms. thick, and the area of each of the opposed

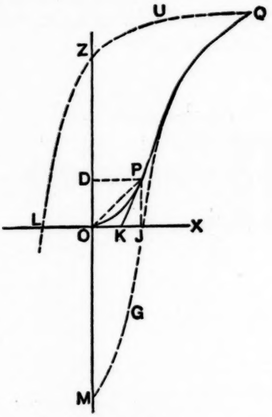

Figure 12.

This illustrates different meanings of the word *inductance*.

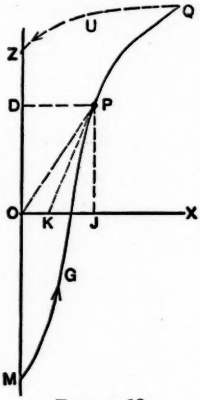

Figure 13.

faces is about 580 square centimeters. The four coils have together 2823 turns, and a resistance at 20° C. of about 12.4 ohms.

Figure 15 shows in outline the electromagnet *Q*, which weighs about 300 kilograms: the core has a square cross-section of about 156 square centimeters area, and is built up, cobhouse-fashion, of soft iron plates about one third of a millimeter thick, each of which was immersed in thin shellac and then thoroughly baked in an electric oven before it was used. Each of the spools, which are practically alike, weighs about 30 kilograms and has four coils, an inner one forming a single layer, the next forming three layers, and the two outer ones wound together side by side from two supply spools, and each equivalent to five layers; in all, both spools together have 3883 turns. The whole core frame is about 74 cms. long and 62 cms. broad. One stratum 2.5 cms. high

and reaching across the middle of the core (Figure 16 *a*) within one of the spools, is made up of five portions insulated from one another, and each of these is surrounded by an exploring coil of insulated wire.

Figure 16 *b* shows the form of the cross-section of the rectangular core frame of a 15 kilowatt transformer (*R*) constructed for experimental purposes and belonging to the Lawrence Scientific School. Besides a low-resistance primary coil, this transformer has 19 similar coils each of about 85 turns, any number of which may be connected to form a secondary circuit. The outside dimensions of the core frame are about 78 cms. and 34 cms.; the area of the cross-section of the finely divided core is about 108 square centimeters.

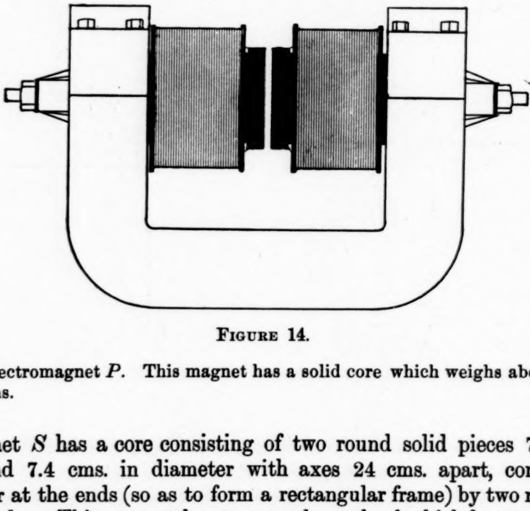

FIGURE 14.

The electromagnet *P*. This magnet has a solid core which weighs about 1500 kilograms.

Magnet *S* has a core consisting of two round solid pieces 76 cms. long and 7.4 cms. in diameter with axes 24 cms. apart, connected together at the ends (so as to form a rectangular frame) by two massive iron blocks. This magnet has two spools, each of which has two coils formed by winding two strands side by side; the whole number of turns is 1724.

The core of magnet *T* forms a square 58 cms. long on the outside and 53.5 cms. wide. Its cross-section is a rectangle 7.5 cms. by 6.7 cms. The core is built up of sheet metal 0.38 of a millimeter thick.

Through the kindness of Dr. George Ashley Campbell I have been allowed to use also seven toroidal coils (of inductances between 0.3 and 13 henries) wound on cores made of very fine (No. 38 B. & S.) iron wire. Such cores are, of course, extremely expensive, but the disturbing

effects of eddy currents in them are practically negligible for the purposes of this paper.

The Demagnetizing of the Core of a Large Electromagnet.

In order to be able to study satisfactorily the magnetic properties of a given piece of iron or steel, it is usually necessary that one should know with some accuracy the magnetic state of the specimen at the outset, and, especially when the metal has the form of a closed ring or frame, the previous history of which is unknown, the only safe pro-

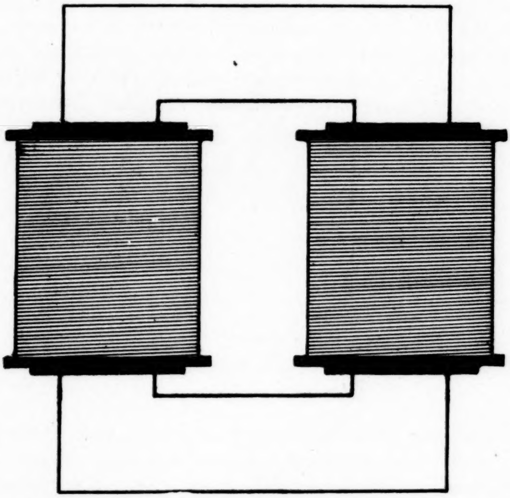

Figure 15.

The electromagnet Q, which has a laminated core made of sheet iron one third of a millimeter thick and weighs about 300 kilograms.

cedure is to demagnetize the iron as completely as possible before one makes any experiments upon it. If the metal has the form of a long rod in a solenoid, or of a slender ring wound about uniformly with insulated wire and magnetized in the direction of its circumference, it is easy to send through the coil which surrounds the iron a long series of currents alternately in opposite directions, which, starting with a value that shall subject the core to a magnetic field at least as strong as any to which it has been previously exposed, gradually de-

crease in intensity to zero. One common way of doing this is to attach the coil to the secondary of a sufficiently powerful alternate current transformer so arranged that the primary coil may be slowly withdrawn to a long distance from the secondary. In the case of the soft iron wire the demagnetization is sometimes accomplished by heating the wire red hot.

It is often a matter of considerable difficulty to remove entirely the effects of previous magnetization from the completely closed massive core of a large transformer: even if the source of a current in the exciting coil has a high voltage, several seconds may be required to established the current, and the use of an alternating demagnetizing current in the coil, with any commercial frequency, is barred out. If a powerful storage battery be connected to the exciting coil through a commutator and a suitable "liquid rheostat," one may begin with a sufficiently strong current (I_0) and, after reversing this several times

FIGURE 16.

Forms of the cross-sections of the laminated cores of the electromagnets *Q* and *R*.

by hand, increase a little the rheostat resistance so as to decrease the current slightly, then reverse this weaker current a number of times, and thus proceed until the current is reduced to a very small value; but if the core is very large, the operation may take a couple of hours even if the number of steps is not excessive, and after all, it is not easy to tell whether the work has been successful. If the initial current was strong enough, if the stages were sufficiently numerous and properly spaced, and if the number of reversals at each step was great, one may, of course, expect to find the core pretty thoroughly demagnetized, but to test the matter it is usually necessary to undo what has been accomplished by determining the amount of magnetic flux sent through the core when a current of given intensity (I) is sent through the exciting coil. This amount ought to be the same whether this testing current has the same direction as that of the last application of the large current (I_0) or the opposite direction, and unless one

has a hysteresis diagram for the core obtained by using currents which range exactly between $+I_0$ and $-I_0$ the whole work must be done twice. The determination of the flux changes may be made very conveniently with the help of a fluxmeter, but if the highest accuracy is required, it is better to take an oscillogram of the building-up curves of the current when the core starts from its state of supposed neutrality.

If the core of a large electromagnet is not quite closed, it is comparatively easy to demagnetize the iron almost completely and to prove that this has been done; indeed, if the gap has the proper width, the

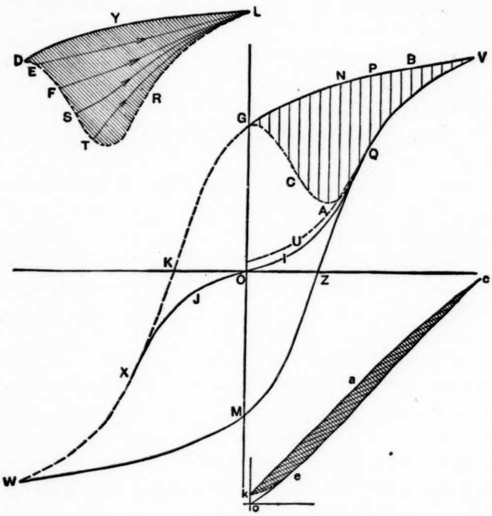

FIGURE 17.

iron practically demagnetizes itself in a wonderful manner. An instance of this was given by Professor Thomas Gray in the case of a 40 K. W. transformer, and I found that the hysteresis diagram for a certain electromagnet which has a solid core the area of which in its slenderest part is more than 450 square centimeters, consists practically of a single straight line when the air gap has a width of 35 millimeters. With this magnet, using an excitation of either 7800 ampere-turns or 15,800 ampere-turns, I obtained current-time curves which were wholly indistinguishable even when much enlarged and

superposed on a screen, whether the current had the same direction as its predecessor or the opposite direction.

If the core of an electromagnet happens to be a straight bar, or a straight bundle of wire, it may be demagnetized by a long series of currents which have alternately one direction and the other, and which slowly decrease in intensity from an initial value which may be considerably smaller than the current which magnetized the iron. Figure 17 shows the results of experiments upon a rod of soft steel 80 diameters long in a long solenoid. The arrangement of the apparatus is shown in Figure 18. The extreme value of the magnetizing field was 27 gausses, and the average moment per cubic centimeter which the

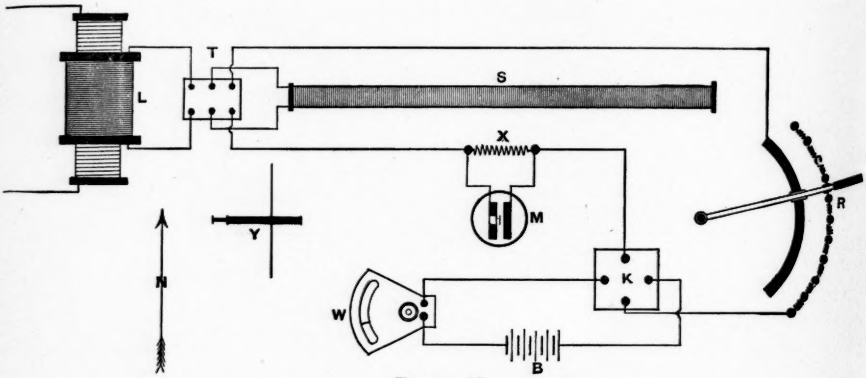

FIGURE 18.

field caused was 246. At the outset the core was thoroughly demagnetized, then a series of steady currents, each a little stronger than the last, was sent through the coil, and the moment of the rod was determined for each direction of the current. This gave the curve $WXOQV$. Then the hysteresis diagram $VGKWMZV$ was obtained, and after the core had returned to the condition indicated by the point V, the current was somewhat decreased until the core "reached" the point B, and then this current was reversed in direction one hundred times, after which (when the current had the positive direction) the iron had exactly arrived at the point on the curve $OIQV$ beneath B. The core was then brought to V again, the current was decreased,— this time until the core reached the point P, — this current was reversed one hundred times, and it was then found that when it ran in positive

direction the core had arrived at the point Q. This process, repeated for many points on the line GPV, yielded the curve $VQACG$. If after being at V the core was brought to a point between P and N, and if after it had been many times reversed the current was decreased by short steps with many reversals at each stage, the core traversed the curve U, whereas if the first drop carried the core no farther than P, the procedure led the core to the origin along the curve I. The lowest point of the curve $VQAG$ lies, of course, nearly over the point Z. The shaded diagram in the upper part of the figure shows a similar curve obtained at another time and drawn strictly to scale. If after many reversals of a comparatively small current the core which started at L reached the point F, and if the current was then slowly increased, the core made the journey indicated by the line FL. The shaded diagram in the lower part of the figure is a reduction of a curve obtained with a large induction coil the core of which is a compact round bundle of fine wire 7.5 cms. in diameter and about 85 cms. long. The curves *oec*, *cak*, *cek*, in this diagram correspond to $OIQV$, VPG, $VQAG$ in the larger figure. The retentiveness of a core of these dimensions is, of course, very small.

Even if much time has been spent in demagnetizing a large closed core by sending through the exciting coil currents alternately in one direction and in the other, of intensities gradually decreasing to a very small final value, it frequently happens that after a much larger current has been put for, say, twenty times through the coil alternately in one direction and the other, the hysteresis cycle does not "close," for the change of flux caused by applying the given current in one direction is not equal to the flux change caused by applying the same current in the other. This fact often makes the accurate determination of a hysteresis diagram for such a core a long and trying piece of work. Some toroidal cores I have never succeeded in demagnetizing completely. The demagnetizing apparatus which I have usually employed in the course of the work here described consists first of a storage battery of forty large cells, a set of rheostats made up of metallic and liquid resistances intended for heavy currents, and a commutator run from the main shaft of the laboratory machine shop, and so arranged as to reverse the direction of the current from the cells every ten seconds. Starting with no resistance in the rheostats, resistance was gradually introduced into the circuit until the current had become very small. After this procedure, the secondary circuit of a specially constructed transformer was attached to the exciting coil of the magnet, and from an initial voltage of about 660, at 60 cycles per second, the electromotive force was gradually decreased until the

current became too small to measure. In some cases it seemed better to omit the second part of the process.

The Establishment of a Steady Current in the Coil of an Electromagnet.

If the circuit of the exciting coil of an electromagnet contains a battery of storage cells of constant voltage E, and if this circuit be suddenly closed, the strength of the current will rise more or less gradually from its initial zero value to E/r amperes, where r is the whole resistance of the circuit in ohms. In the case of a given magnet, with a given electromotive force in the coil circuit, the manner of growth of the current depends very largely, as we have seen, upon the

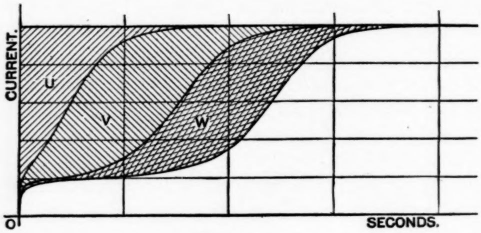

FIGURE 19.

Currents from a battery of 20 storage cells in the circuit of a coil of 2788 turns belonging to the magnet Q. Before the middle curve was taken, the core was carefully demagnetized. The upper and lower curves represent direct and reverse currents, respectively. The areas V and W are equal.

magnetic state of the core when the circuit was closed. The three curves of Figure 19, which are carefully made reproductions of the photographed records of an oscillograph, show the march of the current from a battery of 20 storage cells in the circuit of a coil of 2788 turns belonging to the magnet Q under three different sets of conditions. If after the core had been demagnetized as thoroughly as possible, by the method already described, the circuit was suddenly closed, the current followed the middle curve of the three. If the current was allowed practically to attain its maximum value, and if then a commutator in the circuit was reversed and, at intervals of a few seconds, reversed again and again, and if finally the circuit was broken, it was possible by closing the commutator again in the proper direction, to make the new current follow either the upper or the lower curve of the diagram. If this current coincided in direction with the last current through the

coil, the current was "direct," and its rise was represented by the upper curve. If the new current had a direction opposite to that of the last current through the coil, the current was "reverse," and followed the lower curve. The areas V and W are practically equal.

It is evident that, other things being equal, the rapidity of rise of the current in a circuit which contains a coil wound around the core of an electromagnet will depend very much upon the number of turns in the coil. Figure 20 shows reverse curves from the magnet R. The actual strengths of the currents were 6, 3, and 1.5 amperes respectively, and the numbers of turns in the exciting coils were 85, 170, and 340.

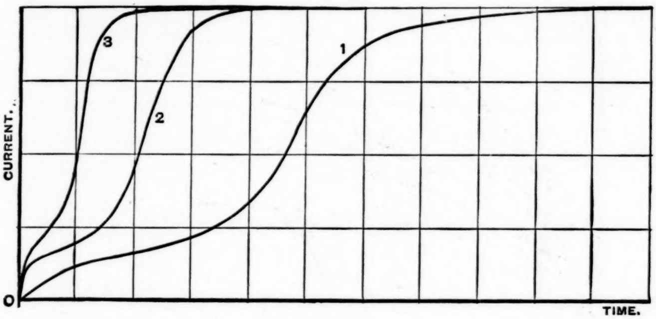

FIGURE 20.

Curves showing the growth of currents in coils of 340 turns, 170 turns, and 85 turns belonging to the magnet R. The same electromotive force was used for all the cases, and the final values of the currents were 6 amperes, 3 amperes, and 1.5 amperes.

The electromotive force was the same in all three cases. The horizontal units are tenths of seconds.

Although the typical current curve for the coil of an electromagnet wound in many turns about the core has two points of inflexion if the core is laminated, both of these disappear if the change of the magnetic flux through the circuit due to the current is small enough, and occasionally one finds an oscillogram which seems to have only one point of inflexion. Some of the direct curves shown in Figures 5, 23, and 28 are everywhere convex upward. Among the nearly three thousand photographed oscillograph records taken for use in this paper no one is concave upward at the very start, but a curve of this kind, with one point of inflexion, has been shown by Dr. Thornton, and I have

many curves which become concave upward very near the origin. In current curves belonging to the coil of an electromagnet which has a large closed, solid core, there are often two points of inflexion, but many of even the reverse curves are everywhere convex upward. Figure 21 shows curves taken for the coil of the large magnet P in the circuit of which was a storage battery of voltage 84. When each current started, the core was nearly neutral.

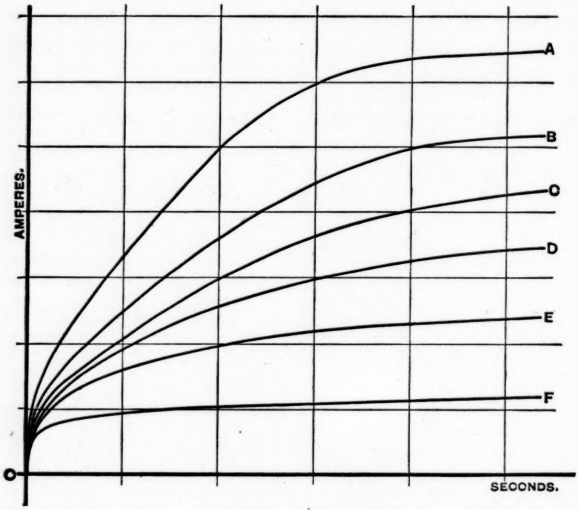

FIGURE 21.

Curves showing the manner of growth of currents of various final strengths in the coil of 2823 turns belonging to the magnet P. The gap was closed and the core was nearly neutral at the beginning of each current. The applied voltage was the same (84) for all the curves.

When the coil of a transformer, the core of which is built up of such thin plates of soft iron as are used in the best practice, is subjected to an alternating electromotive force of extremely high frequency, the disturbing effect of eddy currents in the iron are, of course, very apparent, but the manner of growth of a current under a constant electromotive force is usually not very greatly affected by such currents.

The fact that the susceptibility of the iron is by no means constant, materially alters the shape of a current curve when iron is introduced into a circuit; nevertheless, it is instructive to compare the manner of

growth of a current in the coil of an electromagnet which has such a core, with that of a current in a circuit of fixed inductance, without attempting at the outset to account mathematically for the differences, though it will be easy to do so later on.

In the case of a simple circuit, without iron, of resistance r ohms and *constant* inductance, L henries, which contains a constant electromotive force of E volts, the rise of the current I when the circuit is suddenly closed follows the law

$$I = \frac{E}{r}\left(1 - e^{-\frac{rt}{L}}\right), \tag{16}$$

and attains the fractional part k of its final value (E/r) in the time

$$t = -\frac{L}{r} \cdot \log_e (1 - k), \tag{17}$$

which is independent of the ultimate current strength and involves only the time constant (L/r) of the circuit. If the circuit is made

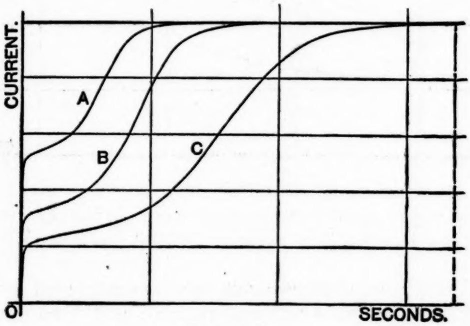

FIGURE 22.

Curves which show the manner of growth of currents in a coil of 1394 turns belonging to the magnet Q, to a given final value, when the applied voltages were 82, 41, and 20.5, nearly. In each case the core was neutral at the outset.

up partly of non-inductively wound resistance wire, and partly of helices, r may be kept constant, while L is changed, by changing the relative proportions of the two parts; or r may be altered while L is constant, by increasing or decreasing the non-inductive portion of the circuit.

If E/r and L are given, different values of E may be used by giving properly corresponding values to the non-inductive resistance, and if the "building-up time" of the current under given initial conditions in the core be defined as the number of seconds required for the current to attain any arbitrarily chosen fractional part of its final value, this time will be inversely proportional to E. In the case of a circuit which has one or more iron cores the phenomenon is much less simple, and if the cores be of solid metal, the effects of eddy currents may complicate the problem seriously; but although under these circumstances the law of proportionality no longer holds, it is almost universally true that the establishment of a current of given final intensity

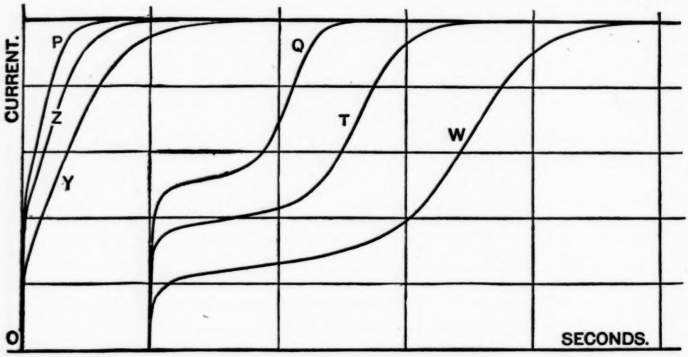

FIGURE 23.

Direct and reverse current curves for the magnet Q with a given final excitation of 2650 ampere turns, under applied voltages of 82, 41, and 20.5, nearly.

in the coil of a given electromagnet can be accelerated by increasing very much the applied electromotive force and then introducing a sufficient amount of non-inductive resistance to make E/r the same as before.

Figure 22 shows current curves for the magnet Q under a fixed final excitation of 2650 ampere-turns. In curves A, B, C, the currents were caused by 40 cells, 20 cells, and 10 cells, respectively, and these currents were made equal by adding to the circuit in each case a suitable non-inductive resistance. Before each of these curves was taken, the core of the magnet was carefully demagnetized by the elaborate process described above. After the magnet Q had been put a good many times through a cycle with a given maximum excitation

of 2650 ampere turns, under one of the voltages just named, direct and reverse curves were taken with the help of the oscillograph. Careful reproductions of these curves are given in Figure 23: to avoid confusion the reverse curves are drawn from a separate time origin.

If in a circuit which contains no iron, E and r be kept constant, while L is changed, the building-up time as defined by equation (17) will be proportional to L. Of course no such simple relation holds when the circuit includes the magnet Q; Figure 24 shows current curves for the same final value of 2.60 amperes, under an applied elec-

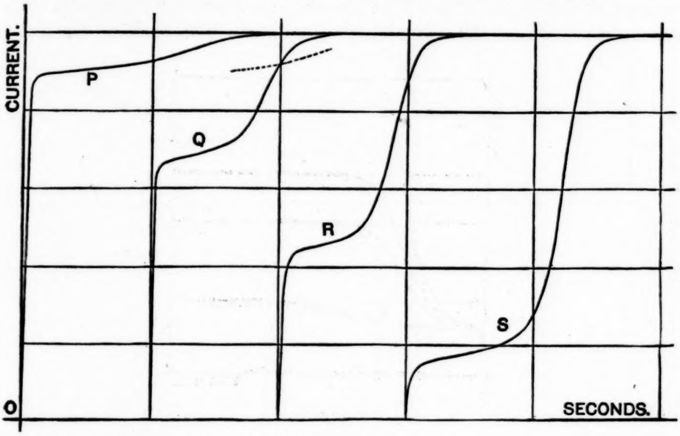

FIGURE 24.

The manner of establishment of a current of final strength 2.60 amperes, in the coil circuit of the magnet Q, under a voltage of 82, when the number of active turns was 407, 823, 1394, or 2788.

tromotive force of about 82 volts, for exciting coils of 407 turns, 823 turns, 1394 turns, and 2788 turns. For convenience, the curves are drawn from different time origins. The dotted line which crosses curve Q calls attention to the fact that if curves P and Q were drawn from the same origin, the former would cross the latter.

If in a circuit without iron E and L were kept constant while r was varied, the building-up time (L/r) would be inversely proportional to the resistance of the circuit, or, since the electromotive force is fixed, directly proportional to the current strength. There is no approximation to this in a circuit which contains iron. The current curves shown in Figure 25 were obtained from the electromagnet Q when

2788 turns were used in the exciting coil and a battery of 40 storage cells with a voltage of about 82 furnished the electromotive force. Curve C evidently corresponds to a case where the total resistance in the circuit is about twice as great as in the case represented by A, but for every value of k the building-up time is greater for C than for A, though the difference becomes very small at the end. A comparison between A and D shows the same fact. Before each of the curves A, B, C, D, was taken the core of the magnet was carefully demagnetized. Figure 26 exhibits current curves taken for different values of r with the same coil of the magnet Q and with the same electromotive force as the curves just mentioned. In each of the cases

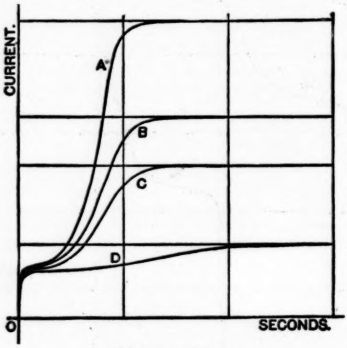

FIGURE 25.

Currents in the coil of the electromagnet Q for four different values of r when E and the number of magnetizing turns were fixed. At the starting of each current the core was magnetically neutral.

shown in Figure 26 the core was put several times through a cycle before the direct and reverse oscillograms were taken. The records are reproduced as accurately as possible; B, C, and D run together in a complicated manner, and the same tendency is shown in the reverse curves G, H, I, but in general the longer building-up times belong to the lower currents.

If in an inductive circuit without iron r and L are fixed, the building-up time will be independent of the value of E, but this is not the fact if the circuit contains an electromagnet. Figures 27 and 28 show current curves obtained from the coil of 2788 turns belonging to the magnet Q. In all the curves of each diagram the value of r was the same, but the voltage of the battery in the coil circuit had three differ-

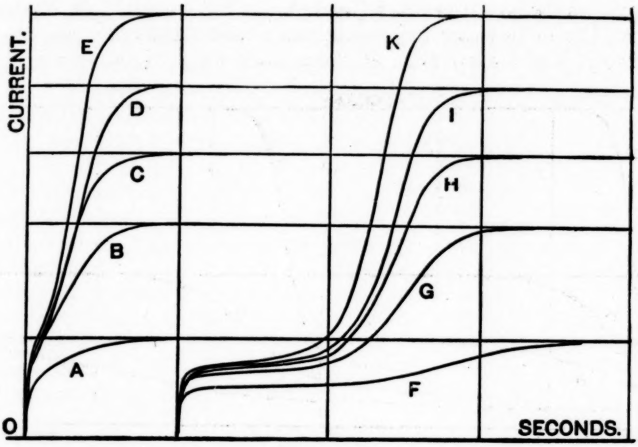

FIGURE 26.

Direct and reverse current curves in the coil of the electromagnet Q for five different values of r when E and the number of active turns were kept fixed.

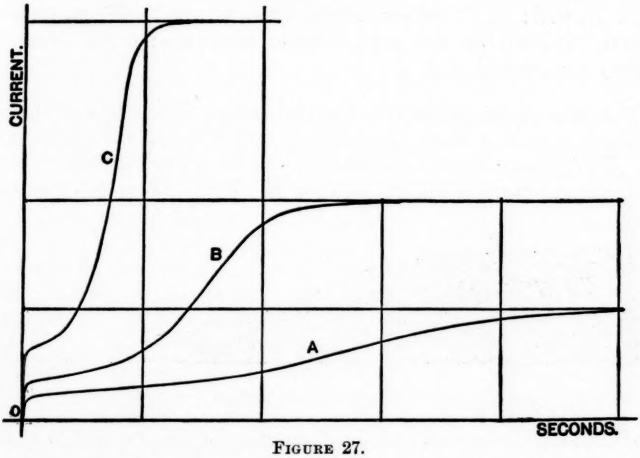

FIGURE 27.

Currents in the coil of 2788 turns belonging to the magnet Q for three different values of the applied voltage with the same value of r. At the starting of each current the core was magnetically neutral.

ent values the largest of which (belonging to the curves C, M, N) was about 82 : in this case the current was almost exactly 2.50 amperes. Before each of the curves A, B, C was taken the core was thoroughly

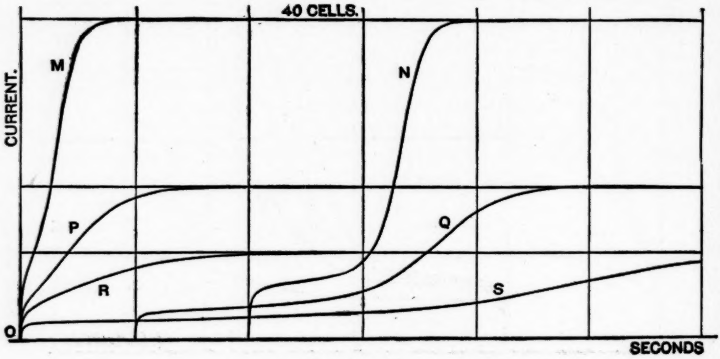

FIGURE 28.

Direct and reverse currents in a coil of 2788 turns belonging to the magnet Q for three different values of the applied voltage, but the same value of r.

demagnetized : R, P, M are direct curves, but S, Q, N are reverse curves. It is evident that the building-up times are not even approximately independent of E.

Figure 29 shows the records of an oscillograph in a secondary circuit in which were a few turns of wire wound around the core of the magnet Q. The primary circuit contained, besides the storage battery, a rheostat and an exciting coil of 1394 turns. When the primary circuit

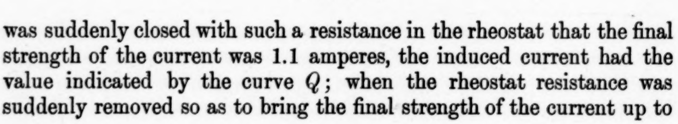

FIGURE 29.

was suddenly closed with such a resistance in the rheostat that the final strength of the current was 1.1 amperes, the induced current had the value indicated by the curve Q; when the rheostat resistance was suddenly removed so as to bring the final strength of the current up to

2.3 amperes, the induced current curve was R. The sum of the areas under the curves Q and R was 74.3 square centimeters. The curve P shows the current record in the secondary circuit when the primary circuit was suddenly closed with no resistance in the rheostat: the area under this oscillogram was 74.6 square centimeters. All the currents were reverse currents. Most of the area determinations of this paper were made with a Coradi "Grand planimètre roulant et à sphère."

Figure 30 shows a careful reproduction of the record of an oscillograph in the primary circuit of the arrangement just described. These curves were taken on the same day as those of the last figure. In this case the flux change due to the current which gave the curve T was to the sum of the flux changes caused by the partial currents as 1130 to 1126. These numbers do not show any real difference between the corresponding physical quantities, but point to difficulties of measurement.

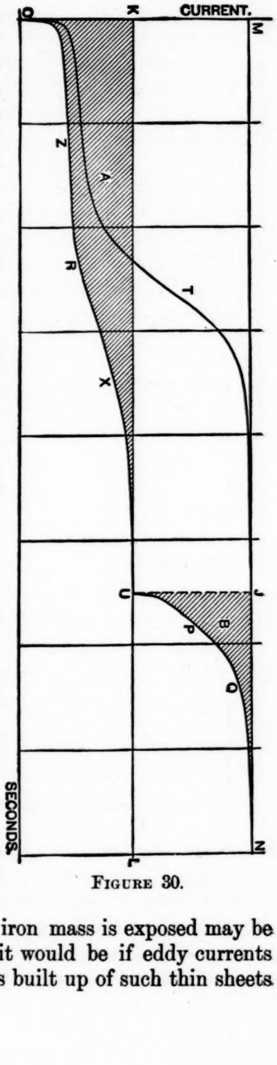

FIGURE 30.

The Effect of the Magnetic Characteristics of the Core upon the Manner of Growth of a Current in the Coil of a Large Electromagnet.

If under the application of a constant electromotive force to the coil circuit of an electromagnet a current grows gradually in the coil to its full value, the magnetic flux in the core at any moment depends, as we have seen, not only upon the instantaneous strength of the current, but also upon the magnetic state of the core at the beginning. Moreover, if the core is solid, it is clear that the magnetizing field to which the interior of the iron mass is exposed may be quite different at any instant from what it would be if eddy currents were nonexistent. If, however, the core is built up of such thin sheets

of iron as are used in good transformers, a fair approximation to the form which the current curve will have under any given circumstances can be made if one has an accurate statical hysteresis diagram of the core for the range required, and if the core is made of very fine varnished wire, as in the case of loading coils for long telephone circuits, a hysteresis diagram obtained either from a long "step-by-step series" of measurements or from one or more oscillograms, enables one to predict with accuracy what the form of a current curve will be for any practical case. These last statements are based on experiments such as those recorded below.

As a result of a long series of measurements, it appears that when the core of the magnet Q has been well demagnetized and a series of steady currents each a little stronger than the preceding one are established in the exciting coil, the magnetic flux through the core in thousands of maxwells follows fairly accurately the course indicated in the following table:

TABLE I.

Ampere Turns.	Magnetic Flux.	Ampere Turns.	Magnetic Flux.
100	35	1100	1208
200	146	1200	1238
300	386	1300	1262
400	622	1400	1285
500	787	1500	1309
600	929	1600	1331
700	1013	1700	1352
800	1086	1800	1369
900	1137	1900	1390
1000	1176	2000	1409

Figure 31 reproduces the table graphically in the full curve: the vertical unit is a thousand maxwells, and the horizontal unit is 139.4 ampere-turns, to suit the case when the particular exciting coil used has 1394 turns. The ordinates of the dotted curve represent twice the corresponding values of the slope (λ) of the other. A template of the curve B was made as accurately as possible from a large piece of sheet

zinc; this was fastened down on a table over a number of sheets of coordinate paper, and the value of λ was determined by measuring on the paper the position of a straight edge which touched the template at any desired point.

TABLE II.

Current in Amperes.	Log $[(13.94)\lambda]$.	Current in Amperes.	Log $[(13.94)\lambda]$.
0.00	0.445	0.55	1.135
0.05	0.860	0.60	1.025
0.10	1.248	0.65	0.943
0.15	1.602	0.70	0.860
0.20	1.715	0.75	0.797
0.25	1.672	0.80	0.746
0.30	1.594	0.90	0.700
0.35	1.496	1.00	0.635
0.40	1.399	1.10	0.621
0.45	1.312	1.25	0.606
0.50	1.209	1.30	0.591

If after the core of Q had been demagnetized, a steady electromotive force of E volts were applied to the exciting circuit of resistance r ohms, containing the coil of 1394 turns, and if eddy currents were nonexistent so that the core followed the statical magnetizing curve, the march of the current (in amperes) would be given by the equation

$$E - ri = 13.94\,\lambda \cdot \frac{di}{dt}, \tag{18}$$

whence

$$t = \int_0^i \frac{13.94\,\lambda}{E - ri}\,di. \tag{19}$$

If from an actual current curve obtained from Q for a given journey of the core we were to determine the corresponding magnetizing curve for the metal (flux versus coil current), we should find that the values of the flux, for small values of the current, at least, would fall short of the flux values which the same currents would cause if they were to act

for some time because the magnetizing field is less than that due to the coil current by that due to the eddy currents. If, therefore, from the numbers of Tables I and II we were to determine the form of a current curve for Q, corresponding to any journey of the core, this would fall somewhat below the actual curve at the beginning. The core of Q has, however, a typical magnetizing diagram, and the theoretical curves are instructive as showing what the actual curves would be if the same core were more finely divided. The effect of eddy currents can be seen in the curves for this magnet given above.

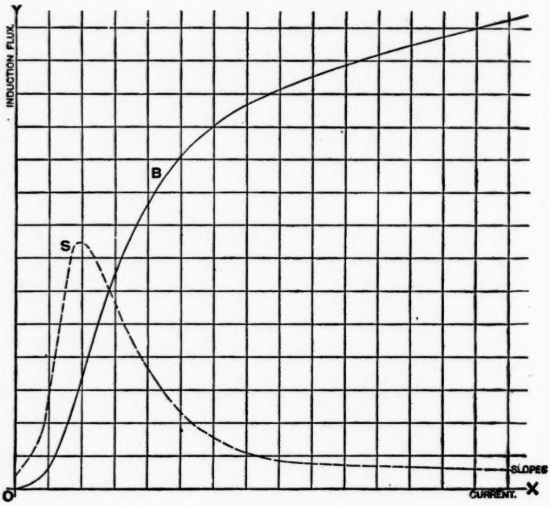

FIGURE 31.

Magnetization curve for the core of the magnet Q which at the outset is in a neutral state. The ordinates of the dotted curve represent twice the slopes of the other curve.

The boundary of the shaded area in Figure 32 shows twice the value of the integrand

$$w = \frac{13.94\,\lambda}{E - ri} \tag{20}$$

for the case $E = 26$, $r = 20$: the horizontal unit is one tenth of an ampere. The vertical line corresponding to $i = 1.3$ is evidently an asymptote. The area under the curve from the beginning to the ordi-

nate representing any given value of the current shows, in twentieths of a second, the time required, under the given conditions, after the circuit is closed for the current to attain this value. It is easy to determine a series of such areas with the help of a good planimeter, and the full curve of Figure 32 actually represents the growth of the current in the case mentioned according to my measurements of the large diagram of which Fig. 32 is a very much reduced copy : for this curve the horizontal unit is one tenth of a second and the vertical unit is one fifth of an ampere. This curve has the general form of most of the

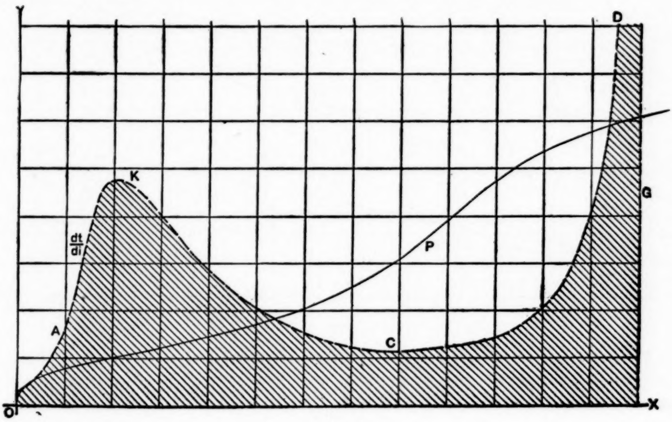

FIGURE 32.

The ordinates of the boundary of the shaded area represent $2(dt/di)$ for $E = 26$, $r = 20$. P shows the theoretical form of the corresponding current curve.

current curves which one obtains with a transformer the core of which is at the outset neutral, but it is evident that in any case where the final value of the current is small enough the asymptote will be moved so far to the left that the integrand curve will rise continually from the beginning, without the maximum and minimum values, and the current curve will have the everywhere convex shape that we find in practice when we cause the current to grow by short steps in the manner indicated by the curve U in Figure 4.

Figure 33 shows building-up current curves (A, b, c) for $E = 26$, and $r = 20$, 40, and 60, respectively. The dotted curves B and C are copies of b and c with ordinates so magnified that the curves have the

same asymptote as A. According to this diagram the current attains 75 per cent of its own final value more quickly when r is 40 than when r is 20, but B crosses A at the point x and the current seems to reach practically its full strength sooner in the latter case. The curve C first crosses the curve A and then B. It would be easy to show from a series of oscillograph records for similar cases that the characteristics of the theoretical curves correspond in general to fact.

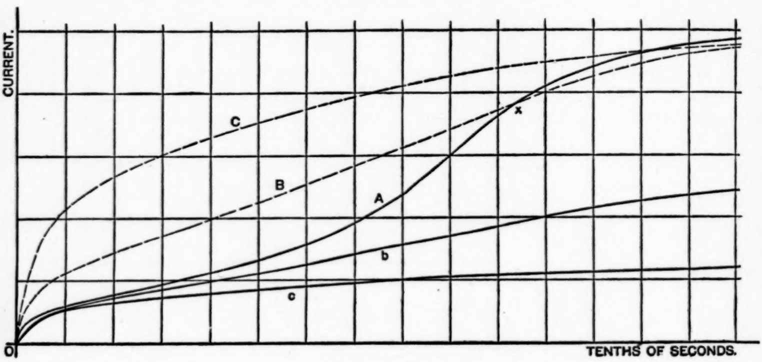

FIGURE 33.

Forms of current curves for Q deduced from theoretical considerations. The coil has 1394 turns and contains a storage battery of voltage 26. C is everywhere convex upward: A and B have two points of inflexion.

If with the core of the magnet Q initially neutral a steady current of given strength be established in the coil of 1394 turns, by use of a storage battery of voltage E, the integrand will be for every value of the current inversely proportional to E (since E/r is given), and the building-up time will be inversely proportional to the applied electromotive force, as it would be if the inductance were fixed. For a given exciting coil, the general shape of the curve for a given current is independent of the applied voltage. Curves A, C, and D of Figure 34 are the current curves computed for $E = 26, 52, 104$, and $r = 20, 40$, and 80: the maximum value of the current is the same in every case. G and F are the current curves computed for $E = 26$, $r = 80$, and for $E = 104$, $r = 320$.

As has been explained already, it is difficult to obtain an accurate hysteresis diagram for a very large core by the ordinary ballistic methods with such galvanometers as are usually to be found in the

testing room, but it is fairly easy to attach extra weights to the suspended system (Figure 35) of a good d'Arsonval or Thomson Mirror galvanometer which shall so increase the moment of inertia that the time of swing shall be lengthened to five or ten or twenty minutes. With an instrument thus modified it is usually possible, by changing the intensity of the current in the exciting coil by small steps, to deal satisfactorily with very large masses of iron. It is of course desirable to use a rather high electromotive force in the exciting coil in order

TABLE III.

Ampere Turns.	Flux in Thousands of Maxwells.	Ampere Turns.	Flux in Thousands of Maxwells.
1812	1371	–131	772
1394	1351	–148	734
1255	1340	–181	552
1031	1316	–234	332
809	1285	–294	22
474	1211	–392	–465
392	1186	–474	–661
294	1148	–809	–1010
234	1121	–1031	–1128
181	1099	–1255	–1214
148	1070	–1394	–1265
131	1060	–1812	–1371
000	953		

to make the building-up time short, and to reduce the current to the desired strength by introducing extra non-inductively wound resistance into the external circuit. In order to test this matter thoroughly, I measured with great care, by aid of a modified Rubens-du Bois "Panzer Galvanometer," the flux changes in the core of the magnet Q (the area of the cross-section of which is more than 150 square centimeters), corresponding to a hysteresis cycle for an excitation of 1812 ampere turns. I then determined the same total flux change by means of planimeter measurements of the areas under a long series of

oscillograph records; all the testing instruments were different in the two cases, and no comparison was possible until the final results were

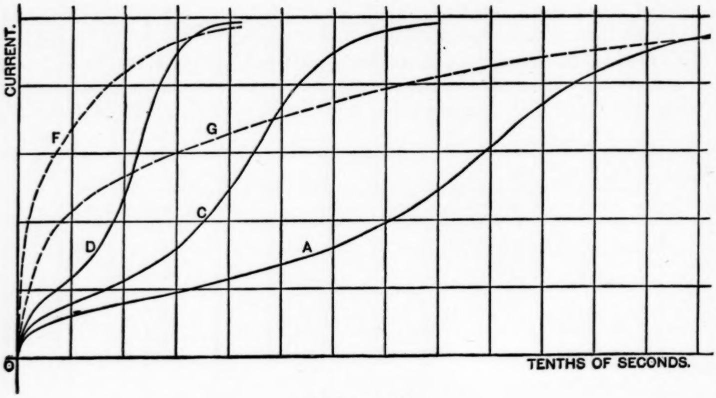

FIGURE 34.

Theoretical forms of current curves in a coil of 1394 turns belonging to the magnet Q. In practice these would be somewhat modified by eddy currents.

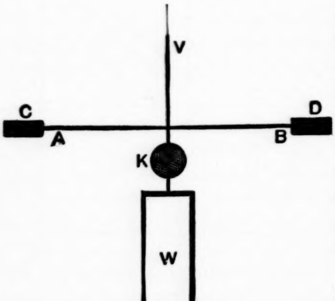

FIGURE 35.

The horizontal rod AB is threaded and the brass masses C, D can be screwed on the rod as far as is necessary. The system must be accurately balanced.

obtained and were found to differ from each other by only one part in about fourteen hundred. The labor of reducing the oscillograms was very great, and this extremely close agreement must be considered accidental, since it is not easy to make a large mass of iron go over exactly the same magnetic journey twice.

Hysteresis diagrams for the magnet Q and corresponding to maximum excitations of 1812, 5370, and 10,880 ampere turns are given in Figure 36. Some results of measurements of the flux changes in the core for the first of these cycles are given in Table III.

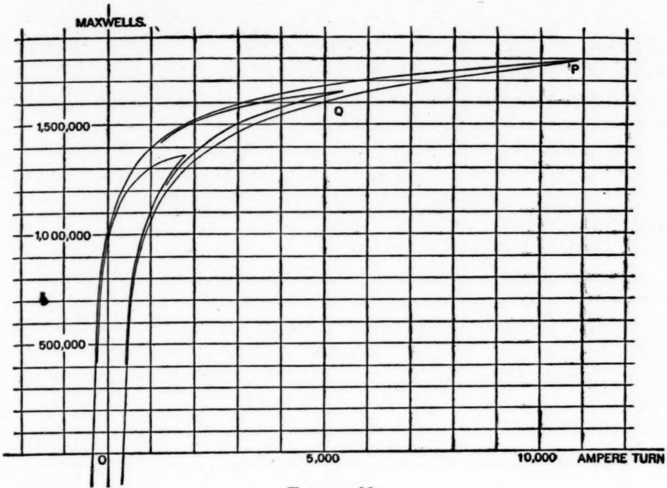

FIGURE 36.

Hysteresis diagrams for the core of the magnet Q.

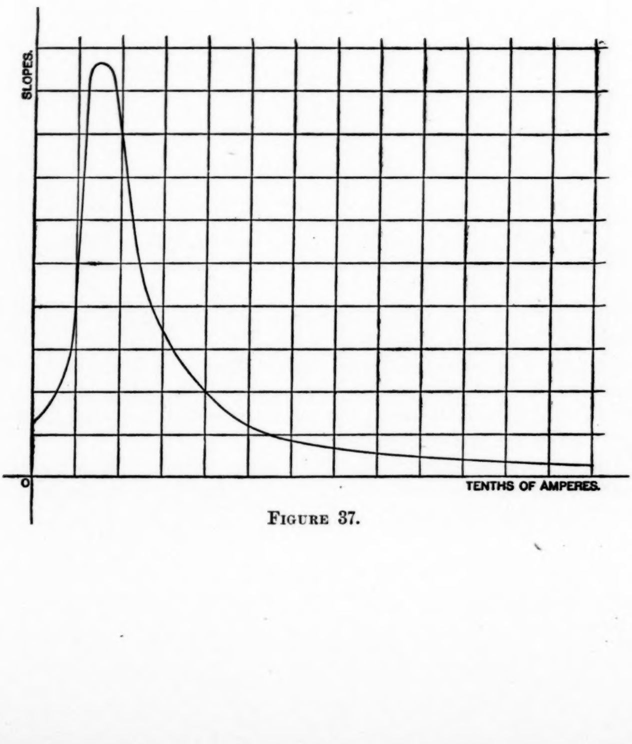

FIGURE 37.

After a curve had been drawn on a very large scale to represent the numbers of Table III, a zinc template was made from it, by aid of which and a long "straight-edge" the slopes of the curve could be determined with some accuracy. The next diagram (Figure 37) shows the slope as a function of the strength of the current.

When the slope for any point of the curve is multiplied by $(13.94) / (E - ri)$, where E and r are given, the result is the value of dt/di for the reverse current curve when the applied voltage is E and the resistance r, for the given value of i. Figure 38 exhibits dt/di for $E = 19.5$, and $r = 15$.

The actual curve was drawn on a large scale, and the area X from $x = 0$ to $x = i$, for a number of different values of i were measured by a planimeter in terms of the unit square of the figure; this area expressed in tenths of seconds the time required for the reverse current to attain the strength i. A few values of X are shown in the next table.

TABLE IV.

i.	$X/10$.	i.	$X/10$.
0.05	0.057	0.50	1.750
0.10	0.155	0.60	1.875
0.15	0.494	0.70	1.985
0.20	0.878	0.80	2.088
0.25	1.141	0.90	2.188
0.30	1.325	1.00	2.294
0.35	1.471	1.10	2.412
0.40	1.579	1.20	2.632

Every form of current curve which I have met in practice can be closely imitated by a theoretical curve; but all these curves have at the outset a direction differing widely from the horizontal. Dr. Thornton, however, shows a beautiful curve which at the beginning is convex downward and has at the start a direction not very different from that of the axis of abscissas.

Before one uses an oscillograph for purposes of accurate measurement, one must make sure that the instrument has been properly set up. When the drum which carries the sensitive film or paper is at

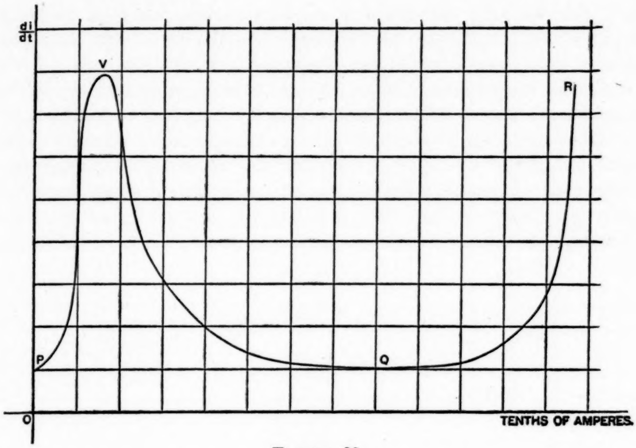

FIGURE 38.

The value of dt/di for a reverse current in a coil of the magnet Q when $E = 19.5$ and $r = 15$.

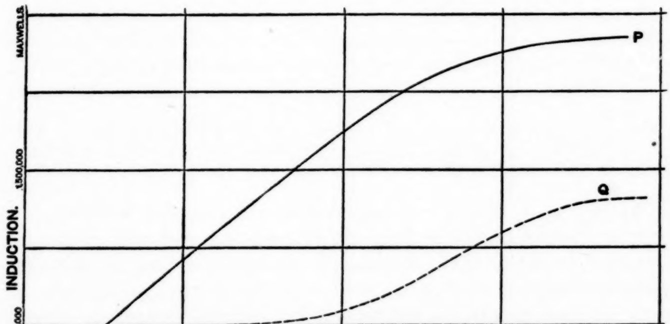

FIGURE 39.

The full curve shows the rate of increase of the flux of magnetic induction through the core of the magnet Q while a reverse current of 1.3 amperes is being established in the exciting coil of 1394 turns. The current curve is shown on an arbitrary scale by the dotted line.

rest, a current sent through the conductor should give a perfectly straight record accurately perpendicular to the base line, and the length of this record should be proportional to the strength of the current. It sometimes happens that an oscillograph which records accurately the march of a moderate current lags in its indications a very little behind the strength of a comparatively feeble current owing to the viscosity of the oil used for damping, which only then becomes troublesome. I have myself had sad experience in drawing from the records of an instrument of this sort, which I thought I had carefully calibrated, elaborate inferences which were contrary to fact. If, however,

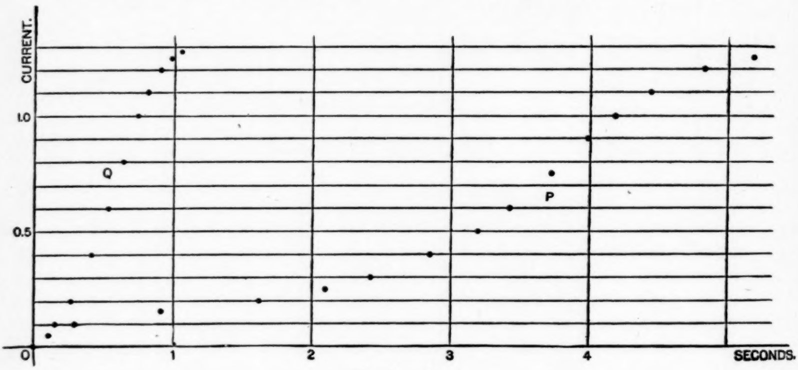

FIGURE 40.

Theoretical forms of direct and reverse current curves for a coil of 1394 turns belonging to the magnet Q when the resistance of the circuit is 8 ohms and the applied voltage is 10.4.

one has at hand, first, a well-constructed and mounted ballistic galvanometer with a period of from eight to ten minutes, and means of damping the swings of the suspended system (electromagnetically or otherwise) without touching it, and secondly, some kind of chronograph designed to close and after a given interval to open again any circuit to which it may be attached, it is easy to test almost any supposed fact about the growth of the flux through the core of an electromagnet.

The toroids I used had cores made of extremely fine, varnished iron wire, costing about four dollars per kilogram. For some of these I determined by ballistic methods, as carefully as I well could, the hysteresis diagrams for several excitations, and then compared with these other diagrams obtained from the oscillograph records of current curves for

the same magnetic journeys of the cores, but I could not detect any differences which did not lie far within the small uncertainty which the viscosity of the oil in the oscillograph may be supposed to cause. It does not seem worth while to print a long series of numbers to illustrate this kind of comparison though the labor was great.

If, then, the core of an electromagnet is made of iron wire not more than one tenth of a millimeter in diameter and carefully varnished, it seems to be true within the limits of accuracy of my measurements and *for the comparatively moderate excitations used*, that if the core

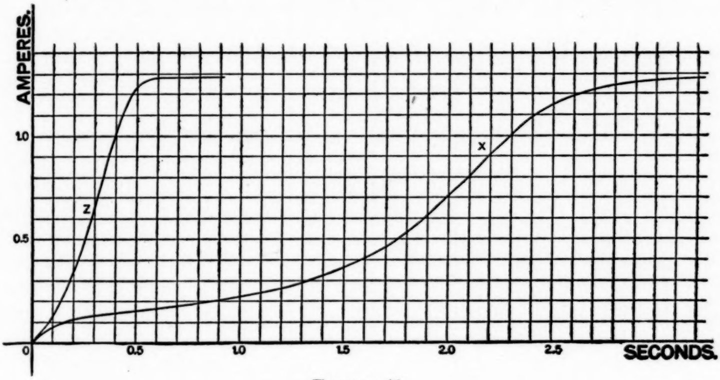

FIGURE 41.

Theoretical forms of direct and reverse current curves for a coil of 1394 turns belonging to the magnet Q when the resistance of the circuit is 15 ohms and the applied voltage is 19.5.

is in a given magnetic state at the start, the change of the flux of magnetic induction caused by a current which grows from zero without decreasing to a given final intensity, is quite independent of the manner of growth of this current. It may grow continuously or by steps, and if eddy currents are not appreciable, the condition of the core at the end is the same. According to this, one would get exactly the same hysteresis diagram from an accurately drawn current curve of the form V (Figure 4) corresponding to any change of current in the exciting coil as from the corresponding U diagram or from any slow step-by-step ballistic method. Nothing of the nature of time lag, if it exists at all, affects the growth of the induction in the iron appreciably. Even in the case of an ordinary transformer, where the effects

of eddy currents are very noticeable at the early portions of most current curves, the whole change of flux due to a given current in the coil is the same apparently whether the current grows steadily or by steps; in this case an accurate diagram of the *U* form and a step-by-step ballistic method with a proper galvanometer may be expected to yield

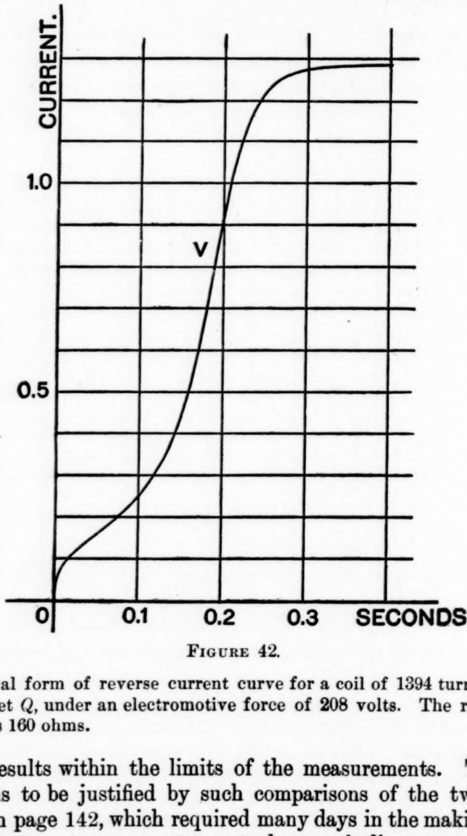

FIGURE 42.

Theoretical form of reverse current curve for a coil of 1394 turns belonging to the magnet *Q*, under an electromotive force of 208 volts. The resistance of the circuit is 160 ohms.

identical results within the limits of the measurements. This statement seems to be justified by such comparisons of the two as that recorded on page 142, which required many days in the making. From a current curve we may expect to get a hysteresis diagram good enough for any commercial purpose, but differing slightly at the beginning from the statical diagram found ballistically. Of course, it would not be easy to get any very accurate information, as some of the curves

given in this paper show clearly, from a current curve taken in the exciting coil of a magnet which has a large solid core.

It will be evident from what precedes that it is possible to predict accurately the building-up curve of a current in the coil of an electromagnet with fine wire core, from a corresponding hysteresis diagram obtained by aid of a ballistic galvanometer of long period, and one of the old methods of procedure.

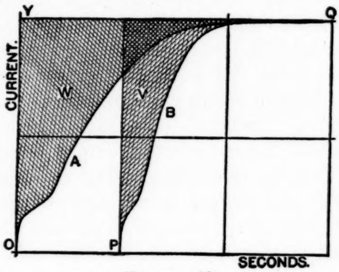

FIGURE 43.

Figure 43 shows two reverse current curves for a toroidal magnet of about one third of a henry inductance belonging to the American Telephone and Telegraph Company. The final strength of the current was the same (1.42 amperes) in both cases, but the applied electromotive force was 10.9 for the left-hand curve and 21.5 for the other. The disturbing effects of eddy currents were here (as will be shown in the sequel) wholly inappreciable. We should be justified in expecting that each of these current curves would yield by aid of a good planimeter a hysteresis diagram substantially the same as any ballistic step-by-step method would furnish for the same magnetic journey of the core.

The Influence of Eddy Currents upon the Apparent Magnetic Behavior of the Core of a Large Electromagnet in the Coil of which a Current is growing.

If after the solid core of a large electromagnet had been demagnetized we were to establish a steady current in the exciting coil by applying to its circuit a constant electromotive force, eddy currents would, of course, be set up in the core, and at any instant during the growth of the current in the coil the iron at the centre of the core would be subjected to a magnetic field weaker than the field belonging to a steady current of intensity equal to the instantaneous strength of the coil current. If, therefore, we were to attempt to determine the magnetic properties of the core from the record of an oscillograph in the coil circuit, we should find that the induction through the core corresponding to a given instantaneous current intensity in the coil was less than the flux belonging to a steady current of the same intensity as deter-

mined from a statical hysteresis diagram. The same phenomenon appears when an electromagnet with finely laminated core has a secondary coil. The closing on itself of a secondary coil wound on the core of an electromagnet when a current is being established in the primary will, therefore, expedite at first the rise of this current, but the area over the current curves ought to be the same in the two cases, and we must expect, therefore, the building-up time to be somewhat longer when the secondary coil is closed than when its circuit is broken.

It is to be expected, of course, that the curves which show the march of the current in the primary circuit will be noticeably different in form when the secondary circuit is closed and when it is open; for this is often the fact in the case of two neighboring circuits which have *fixed* self and mutual inductances (L_1, L_2, M) if one of them containing an electromotive force E be suddenly closed at the time $t = 0$, while the other, which contains no electromotive force, is closed. Here

$$\left.\begin{aligned} L_1 \cdot \frac{dI_1}{dt} + M \cdot \frac{dI_2}{dt} + r_1 \cdot I_1 &= E_1, \\ M \cdot \frac{dI_1}{dt} + L_2 \cdot \frac{dI_2}{dt} + r_2 \cdot I_2 &= 0, \end{aligned}\right\} \tag{21}$$

where r_1, r_2 are the resistances of the circuits and I_1, I_2 the currents in them.

If $$\lambda = -\frac{(Q-R)}{2S}, \quad \text{and} \quad \mu = -\frac{(Q+R)}{2S},$$

where $S = L_1 \cdot L_2 - M^2$, $Q = r_2 \cdot L_1 + r_1 \cdot L_2$, $R^2 = Q^2 - 4 r_1 \cdot r_2 \cdot S$;

$$I_1 = \frac{E_1}{R \cdot r_1}\left[R - \tfrac{1}{2} e^{\lambda t}(r_2 \cdot L_1 - r_1 \cdot L_2 + R) + \tfrac{1}{2} e^{\mu t}(r_2 \cdot L_1 - r_1 \cdot L_2 - R)\right], \tag{22}$$

$$I_2 = \frac{E_1 \cdot M}{R}\left[e^{\mu t} - e^{\lambda t}\right], \tag{23}$$

$$\int_0^\infty I_2 \cdot dt = -\frac{E_1 \cdot M}{r_1 \cdot r_2}, \quad \text{and} \quad \int_0^\infty \left(\frac{E_1}{r_1} - I_1\right) dt = \frac{L_1 \cdot E_1}{r_1^{\,2}}. \tag{24}$$

Figure 44 illustrates a typical case where S is positive: the heavy line shows the current in the primary circuit when $r_1 = 3$ ohms, $r_2 = 2$ ohms, $L_1 = 3$ henries, $L_2 = 2$ henries, $M = \sqrt{6}/3$ henries, $E_1 = 12$ volts, when the secondary is closed; the lighter curve shows the rise of the current in the same circuit when the secondary circuit is open.

$$I_1 = 4\left(1 - \tfrac{1}{2}e^{-\frac{3t}{4}} - \tfrac{1}{2}e^{-\frac{3t}{2}}\right), \tag{25}$$

and

$$I_1 = 4\left(1 - e^{-t}\right). \tag{26}$$

The slope of the first curve is at the outset somewhat greater than that of the secondary curve, but eventually becomes less, the curves intersecting at a point Y. The area between the curve and the asymptote drawn parallel to the axis of abscissas is the same for both cases.

If the circuits just described had in common a large closed iron core, the current curves for open and closed secondary circuit would be

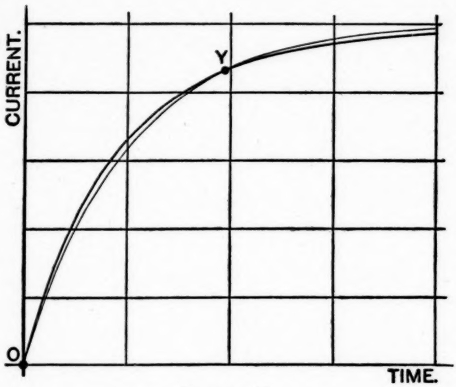

FIGURE 44.

Currents in the primary circuit of an induction coil with air core, when the secondary circuit is closed (full curve) and when the secondary is open.

much less like each other than the curves of Figure 44 are, even if the core were not solid. We may illustrate this fact by some oscillograms from a transformer which has a laminated core.

Figure 45 shows two typical reverse current curves for the exciting coil of the magnet Q which has 2788 turns, when the circuit of a secondary coil of 1095 turns is (D) open and (C) closed. Both curves rise very rapidly at the start, and then bend suddenly, so as to become almost horizontal for a time, but in the first fifth of a second the curve taken when the secondary is closed attains 40 per cent of its final value, and the other curve only 18 per cent; yet the second curve reaches half its height about two fifths of a second sooner than the first does; and when the secondary is open the current in the primary

circuit reaches 98 per cent of its maximum strength in about $\frac{5}{8}$ths of a second less time than when the secondary is closed. In this case the final current was 2.80 amperes. Of course the degree of divergence of the current curve for the primary circuit when the secondary is closed, from the corresponding curve when the secondary is open, depends very much upon the number of turns of the secondary and upon its resistance.

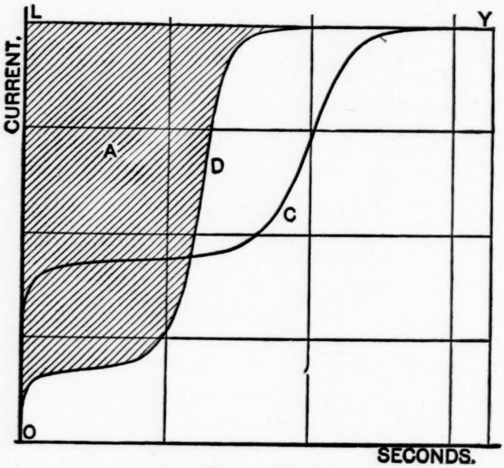

FIGURE 45.

Reverse current curves for the coil of 2788 turns belonging to the magnet Q, when the circuit of a secondary coil of 1095 turns was closed (C) and open (D). The resistance of the primary circuit, which contained a battery of 40 storage cells, was 30 ohms.

Figure 46 shows both reverse and direct curves for the magnet Q when the primary and secondary coils were geometrically alike and each had 1394 turns. The resistance of the primary circuit was about 16.7 ohms.

The curves of Figure 47 belong to a primary coil of 823 turns of the magnet Q. The lines which have O as origin represent currents of about 2.05 amperes due to a storage battery of 10 cells; the lines which start at X were caused by currents of 7.55 amperes from a battery of 40 cells.

Figure 48 shows direct and reverse curves for a current of 3.30 amperes (due to a storage battery of 40 cells) in a coil of 1394 turns

belonging to Q. The curves M, N were taken with a secondary coil of 16 turns and comparatively high resistance closed; the boundaries

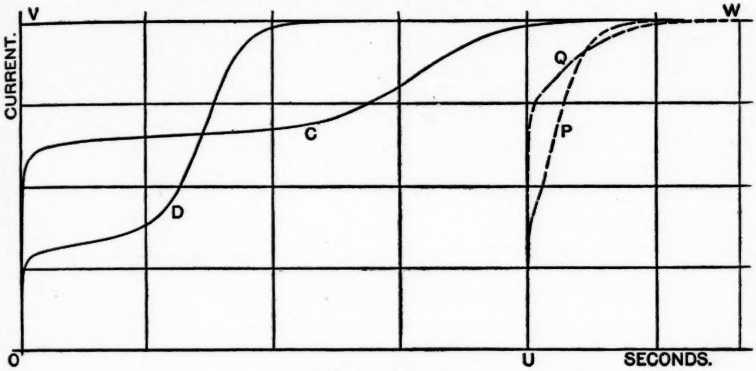

FIGURE 46.

Direct and reverse current curves for a coil of 1394 turns belonging to the magnet Q when a secondary circuit of 1394 turns was closed and open.

of the shaded areas m, n show the forms of the currents induced in this secondary as obtained from an oscillograph in the circuit. Since

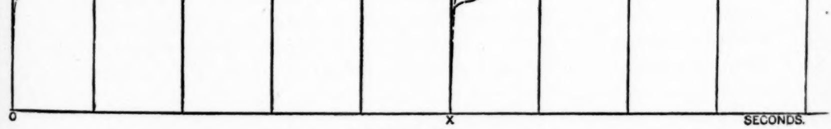

FIGURE 47.

Direct and reverse curves representing currents in a primary coil of 823 turns belonging to the magnet Q, for open and closed secondary circuit. The secondary coil had 2788 turns. For the curves which start at O the voltage was about 20.6; for the curves which begin at X the voltage was about 82 and the maximum current 7.55 amperes.

the number of turns in this secondary was so small and the resistance large, the forms of the curves *M*, *N* are not very different from what they would have been if the secondary circuit had been open. The curves *V*, *W* were taken with another secondary circuit of 1095 turns closed on itself: the boundary of the area *v* shows on an arbitrary scale the form of the induced current in this last mentioned secondary circuit.

It is not to be expected, of course, that a current curve for the exciting coil of an electromagnet which has a large solid core will be so much altered in general appearance by the closing of a secondary coil

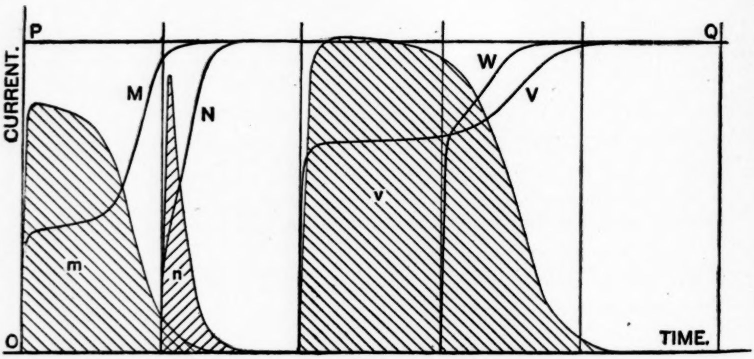

FIGURE 48.

as it would be if the core were divided so as to prevent in large measure the effects of powerful eddy currents which are present when the iron is in one piece.

Even in the case of an electromagnet the core of which is built up of broad varnished pieces of sheet iron, eddy currents in this iron may radically change the form of a current curve unless the sheets are very thin. Figure 49 illustrates this fact by an actual example drawn to scale.

Figure 50 shows curves belonging to a certain transformer. *M* is a piece of a statical hysteresis curve; *N* is a similar curve obtained from a reverse current oscillogram. Although the core of this magnet is made up of varnished pieces of sheet iron, the effects of eddy currents, as will be shown more clearly in the sequel, are here very noticeable.

Some instances of the phenomenon just mentioned suggest a possible

pure time-lag[12] of magnetization, like that observed by Ewing and Lord Rayleigh, large enough in the case of a very large core to affect somewhat the forms of the current curves; in fact, I have spent a very long time and have made many measurements upon a great number of oscillograph records in order to see whether any such lag could be shown; but after all allowances have been made for the effects of eddy currents, nothing tangible, if anything at all, remains, for such moderate excitations as I have used with closed, finely divided cores.

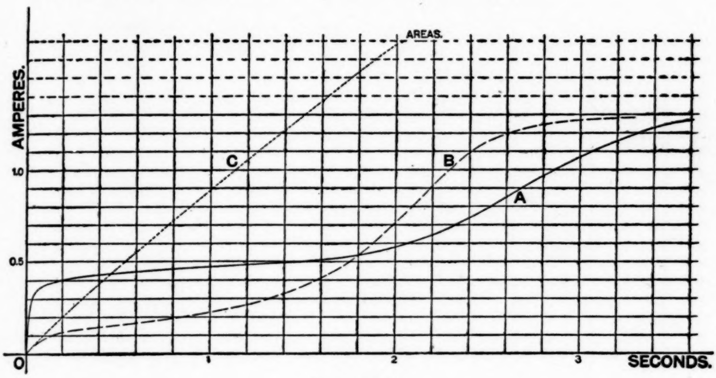

FIGURE 49.

The full line represents the actual form of a reverse current curve in the coil of a certain transformer the core of which is laminated; the curve sketched out by dashes represents the theoretical form as obtained from the statical hysteresis diagram. The dotted curve represents on an arbitrary scale the areas between the real curve and the asymptote; the flux change being nearly proportional to the time.

If to a circuit — without iron and unaffected by any neighboring currents — which has a fixed inductance L, and resistance r, be applied a fixed electromotive force, E, the current-time curve will follow the equation

$$I = \frac{E}{r}\left(1 - e^{-\frac{rt}{L}}\right),$$

and the current will attain the intensity $I_0 = E/(r+h)$ at the time t_0 such that

[12] G. Wiedemann, Galvanismus, **3**, 738. Ewing, Magnetic Induction, § 84. Gumlich und Schmidt, Electrotechnische Zeitschrift, **21**, 1900. Rücker, Inaugural Dissertation, Halle-Wittenberg, 1905.

$$e^{-\frac{rt_0}{L}} = \frac{h}{r+h}.$$

If, however, the resistance of the circuit at the outset had been $(r+h)$ and if after the final value of the current I_0 for this resistance had been established, the extra resistance had been suddenly removed from the circuit, the current curve from that instant on would have followed the equation

$$I = I_0 e^{-\frac{rt'}{L}} + \frac{E}{r}(1 - e^{-\frac{rt'}{L}}),$$

or, since

$$I_0 = \frac{E}{r}(1 - e^{-\frac{rt_0}{L}}),$$

$$I = \frac{E}{r}(1 - e^{-\frac{r(t'+t_0)}{L}}).$$

It is clear, therefore, that in the case of a circuit of this kind the last (upper) portion of a step curve of the form U (Figure 4) will have exactly the same shape as the corresponding part of the V curve, although the lower portions may be very different.

If in the case also of a circuit which has one or more finely divided iron cores the flux of induction through the circuit can be considered as a single valued (given) function of the current strength when the magnetic state of the iron at the outset is given, the upper portion of a curve of the U type (Figure 4) belonging to the circuit will be identical with the corresponding part of a curve of the V type. We need consider only a U curve with one intermediate step. If the induction (N)

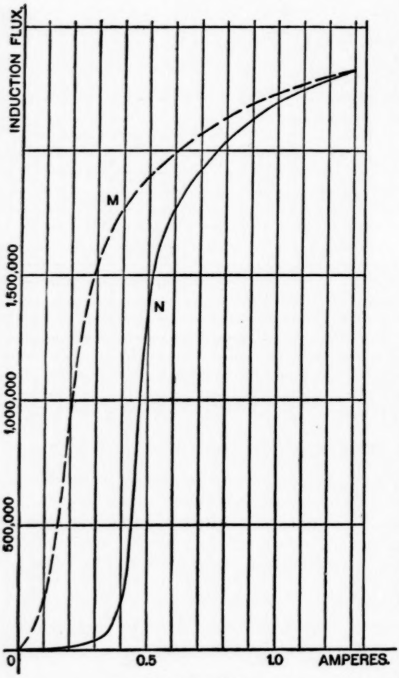

FIGURE 50.

M is a portion of a statical hysteresis diagram for a certain transformer under an excitation of 1812 ampere turns. N is a similar curve obtained from a reverse current oscillogram.

through the circuit corresponding to a current of intensity I is $\phi(I)$, and if the resistance of the circuit is R, the differential equation which determines the growth of the current is

$$E - \frac{dN}{dt} = RI \quad \text{or} \quad \frac{\phi'(I) \cdot dI}{E - RI} = dt.$$

Since ϕ is known, the coefficient of dI is known after values have been assigned to the constants E and R. If with a given E, R has the value r, the curve obtained by plotting the coefficient of dI against I will have a shape something like that of the line $KCDP$ of Figure 51, which has the line $I = E/r$ for an asymptote. If with the same value of the electromotive force R has the value $(r + h)$, the curve will have a shape something like that of the line $KBDA$, which has the vertical asymptote $I = E/(r + h)$ which passes through Q. If with the core in the state for which the diagram is drawn, the circuit be closed at the time $t = 0$, and if the resistance be $(r + h)$, the time required for the current to attain any value I' less than $E/(r + h)$ is proportional to the shaded area under the curve $KBDA$ from the ordinate axis up to the vertical line $x = I'$. If, however, the resistance of the circuit had been r, the time required for the current to grow to the intensity I' would be represented on the same scale by the area under the curve $KCDP$ from $x = 0$, to $x = I'$. If the circuit were closed when its resistance was $(r + h)$, and if the current were allowed practically to reach its final value for this resistance, as represented by the line OE, and if then the resistance h were suddenly shunted out, the current would grow to its new final value at a rate determined by the fact that the time required to reach the current OH must be equal, on the scale of the diagram, to the area $EFPH$. If the circuit had been closed first when its resistance was r, the time required for the current to grow from the intensity OE to the intensity OH would still be equal, on the scale used, to the area $EFPH$, and the shape of the current curve, from $E/(r + h)$ on, would be the same as before. Of course the N of this theory need not be the same as the N of the statical hysteresis diagram for the given magnet; it might

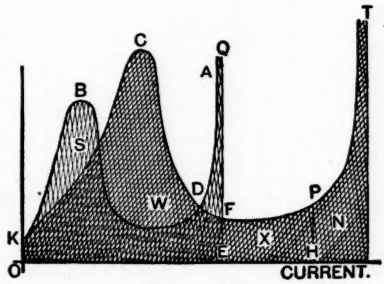

FIGURE 51.

have for any value of I a value which in the case of the statical curve belonged to a current weaker by any given constant or otherwise determined amount. The curve FP must, however, have the same form for a continuously growing current and for one which suddenly begins to increase from the value OE.

As a matter of fact, experiment seems to show that if the core of an electromagnet is made of varnished wire so fine that eddy currents are practically shut out, the upper portion of a U curve with a single intermediate step is exactly like the corresponding portion of the V curve. Figure 52 represents a set of current curves obtained from a number

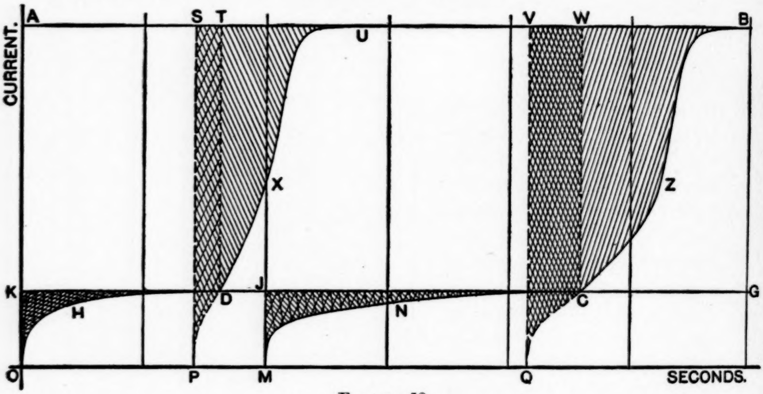

FIGURE 52.

Current curves for a coil with fine wire core. The second part of a two-stage current is exactly the same as if the current were allowed to grow at once to its final value.

of toroidal coils (with very fine wire cores) connected up in series; the current came from a storage battery of ten cells. When the circuit had its normal resistance, the final value of the current was represented by OA; it was possible, however, to close the circuit with such an extra amount of resistance that the final value of the current should be representable on the same scale as before, by the line OK. The extra resistance could then be suddenly shunted out of the circuit by closing a switch at any time after the lower current had practically attained its maximum strength. When the core had been previously demagnetized, a diagram of this kind had the form $OHDXU$; but if the circuit had from first to last its normal resistance, the current curve had a shape accurately represented — when the starting point was shifted to the proper

point (P) on the time axis — by $PDXU$. The upper part of the curve was in no way distinguishable from the corresponding portion of the U diagram. Mr. John Coulson and I have taken many records of this kind and have not been able to detect any difference between the upper parts of the different kinds of curves. The second part of the U diagram starts off at exactly the same angle with the horizontal that the other curve has when the line KG is crossed. The area $OKDHO$ when divided by the length OK should be the same as the area $PSTDP$ divided by the length OA.

If eddy currents are present, the upper portions of a U diagram and of a V diagram do not entirely agree. Figure 53 represents diagrams

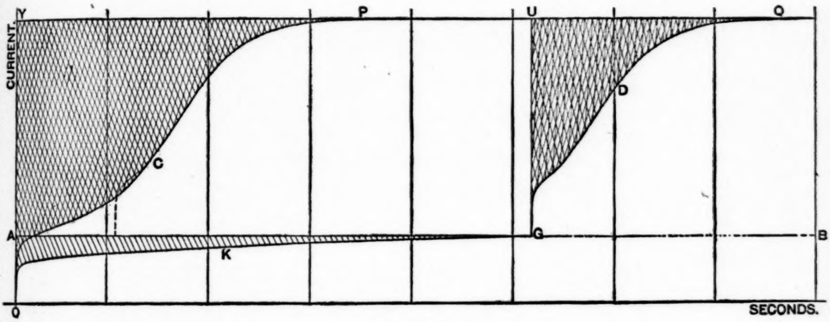

FIGURE 53.

Growth from an originally neutral core of a current in a transformer with a laminated core. The effects of eddy currents are here noticeable.

for the magnet Q which has a laminated core, although eddy currents are not entirely shut out. If the upper part of the U diagram (GDQ) be shifted to the left, it will be found to agree with the curve PCO from P to C, but beyond C the two are quite different, as the dotted line indicates. When the V current, the growth of which is represented by the line OCP, has reached the strength OA, the induction flux through the core is only a small fraction of the flux when a steady current of final strength OA is established in the coil in the manner represented by OKG. The existence of eddy currents is indicated clearly by the fact that the first portion of the curve GDQ is nearly vertical. These diagrams were obtained when the core had been well demagnetized. Figure 54 shows similar diagrams for direct curves (dotted) and for reverse curves (full).

THE GROWTH OF THE INDUCTION FLUX IN THE CORE OF AN ELECTROMAGNET WHILE THE EXCITING CURRENT IS TEMPORARILY CONSTANT.

It sometimes happens that if a number of secondary coils of low resistance, wound upon the core of an electromagnet, are closed on themselves, the building-up curve of a current in the exciting coil is for a comparatively long time almost exactly parallel to the time axis. During this time it is difficult to detect any change in the intensity of the current, and yet the flux of magnetic induction through the core is increasing at a very nearly constant rate. This fact, which has a certain pedagogic interest, is easily illustrated. The curve

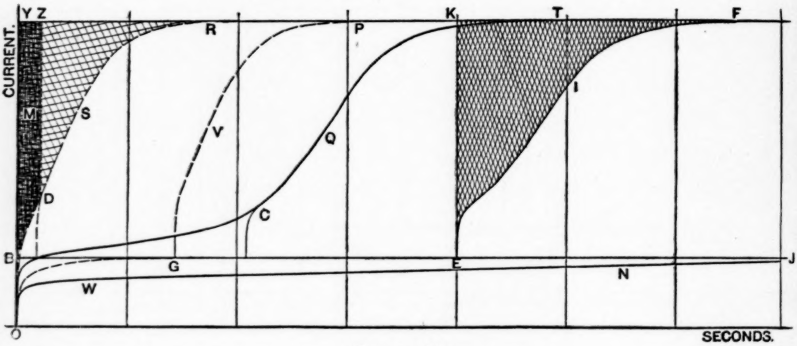

FIGURE 54.

Direct and reverse current curves for a transformer with a laminated core. The existence of eddy currents is clearly shown.

OPQU (Figure 55) shows a nearly typical case, and the line *OKLG* represents on a different scale the induced current in one of the secondary circuits. To a person watching an amperemeter in the primary circuit, the current seems to have attained its final value in less than a second, and if he leaves the instrument at the end of, say, five seconds, he feels sure that the current has become steady. Meanwhile the induction flux, as measured on the scale of the diagram by the area between the curve and the line *YU* (or, on a different scale, by the area under the curve *OKLG*), is constantly growing. Of course if the core is very large, the whole building-up time may be a minute or more, and the phenomenon may then become very striking.

The magnet *T* has three coils. The first (*A*) has 750 turns, the

second (B) 250 turns, and the third (C), which is made of wire of very large cross-section, has a small unknown number. Figure 56 reproduces accurately the records of two oscillographs, one in the coil A, the other in B, when C was closed. $OMQL$ is a part of the building-up curve for the main circuit (A), and $Ocbk$ is a corresponding portion of the record of the induced current in B. In the case represented by the full line $OMQTVW$, the coil C was suddenly opened at about 1.05 seconds after the start: $Ocbznda$ shows the record of the induced current in B under these circumstances. The scales of the two oscillographs were, of course, not the same. The sudden jumps in the oscillograms might have been predicted, in amount as well as in direction, by the principle of the "Conservation of Electromagnetic Mo-

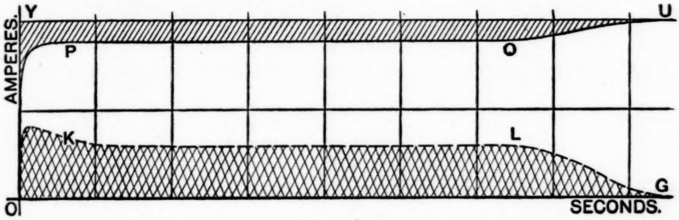

FIGURE 55.

menta." We shall return to the subject of the sudden changes brought about in the currents in inductively connected circuits when the inductances of the system are impulsively changed.

THE EFFECTIVENESS OF FINE SUBDIVISION IN THE CORE OF AN ELECTROMAGNET FOR THE PREVENTION OF ELECTROMAGNETIC DISTURBANCES DUE TO EDDY CURRENTS, WHEN A STEADY ELECTROMOTIVE FORCE IS APPLIED TO THE CIRCUIT OF THE EXCITING COIL.

In order to determine approximately the magnitude of the effect of eddy currents upon the growth of a current [13] in the coil of an electromagnet the core of which is made of fine iron wire, we may consider the case of a very long solenoid consisting of N turns of wire per centimeter of its length, wound closely about a long prism of square cross-

[13] The influence of eddy currents in the formation of a regularly fluctuating current in the exciting coil of a transformer under a given, alternating electromotive force has been studied by J. J. Thomson for cores of square cross-suction built up of iron sheets, and by Heaviside for round cylindrical cores cut radially. See the Electrician for April, 1892, and Heaviside's Electrical Papers, **1**, xxviii.

section ($2a \times 2a$) built up uniformly (Figures 59 and 60) of a large number of varnished filaments of square cross-section ($c \times c$), or else consisting of a bundle of infinitely long straight wires. The axis of the prism shall be the z axis, and the x and y axes shall be parallel to faces of the prism. The electric resistance of the solenoid per centimeter of its length shall be w, the constant applied electromotive force per centimeter of the length of the prism shall be E, and the intensity of the current in the coil shall be C. Within the core, the magnetic field (H) will have the direction of the z axis, and if q is the current flux at any place

$$4\pi q = \text{Curl } H, \tag{27}$$

or

$$4\pi q_x = \frac{\partial H}{\partial y}, \qquad 4\pi q_y = -\frac{\partial H}{\partial x}, \qquad 4\pi q_z = 0.$$

Within any filament of iron in the core, H satisfies the equation

$$\frac{\partial H}{\partial t} = \frac{\rho}{4\pi\mu}\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right), \tag{28}$$

where ρ is the specific resistance of the iron and μ is its permeability, which for the present purpose shall be regarded as having a fixed value.

When there are no Foucault currents in the core, the intensity (H) of the magnetic field within has at every point the boundary value H_S or $4\pi NC$, but if positively directed eddy currents exist, H may be greater at inside points than at the surface. We need not distinguish between the flux p through the turns of the coil per centimeter of its length, and N times the induction flux $\mu \iint H\, dx\, dy$ through the core, so that we may write

$$E - \frac{dp}{dt} = E - \mu N \iint \frac{\partial H}{\partial t} \cdot dx\, dy = w \cdot C = \frac{w \cdot H_S}{4\pi N}, \tag{29}$$

or by virtue of (28),

$$E = \frac{w \cdot H_S}{4\pi N} + \frac{\mu\rho N}{4\pi\mu} \iint \left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right) dx\, dy, \tag{30}$$

where the integration extends over a cross-section of the core.

The vector H is always perpendicular to its curl, and the intensity of the component of the current at any point in the iron, in any direc-

tion, s, parallel to the xy plane at any instant, is equal to $1/4\,\pi$ times the value at that point, at that instant, of the derivative of H in a direction parallel to the xy plane, and 90° in counter clockwise rotation ahead of s.

Along any curve in the iron parallel to the xy plane, H must be constant if there is no flow of electricity across the curve. At every instant, therefore, the value of H at the boundary common to any two filaments must be everywhere equal to H_S. If the coil circuit is broken, H must be constantly zero at the surface of every filament.

Two or three general theorems concerning solutions of differential equations of the form

$$g\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=\frac{\partial w}{\partial z},$$

will be helpful to us.

If v and w represent any analytic functions of x, y, z, and if $L(w)$, $M(v)$ represent the adjoint differential expressions

$$g\cdot\frac{\partial^2 w}{\partial x^2}+g\cdot\frac{\partial^2 w}{\partial y^2}-\frac{\partial w}{\partial z}, \tag{31}$$

$$g\cdot\frac{\partial^2 v}{\partial x^2}+g\cdot\frac{\partial^2 v}{\partial y^2}+\frac{\partial v}{\partial z}, \tag{32}$$

the corresponding form of the generalized Green's Theorem may be expressed by the equation,

$$\begin{aligned}\iiint[v\cdot L(w)-w\cdot M(v)]\cdot dx\,dy\,dz = \\ g\iint\left(v\cdot\frac{\partial w}{\partial x}-w\cdot\frac{\partial v}{\partial x}\right)\cdot\cos(x,n)\cdot dS+ \\ g\iint\left(v\cdot\frac{\partial w}{\partial y}-w\cdot\frac{\partial v}{\partial y}\right)\cdot\cos(y,n)\cdot dS-\iint w\,v\cdot\cos(z,n)\cdot dS;\end{aligned} \tag{33}$$

and it is easy to prove that

$$\begin{aligned}\iiint v\cdot L(w)\,dx\,dy\,dz = g\iint v\left(\frac{\partial w}{\partial x}\cdot\cos(x,n)+\frac{\partial w}{\partial y}\cdot\cos(y,n)\right)dS \\ -g\iiint\left(\frac{\partial w}{\partial x}\cdot\frac{\partial v}{\partial x}+\frac{\partial w}{\partial y}\cdot\frac{\partial v}{\partial y}\right)dx\,dy\,dz-\iiint v\cdot\frac{\partial w}{\partial z}\cdot dx\,dy\,dz.\end{aligned} \tag{34}$$

If w and v are identically equal, the last equation becomes

$$\iiint w \cdot L(w) \cdot dx\,dy\,dz = g \iint w \left(\frac{\partial w}{\partial x} \cdot \cos(x, n) + \frac{\partial w}{\partial y} \cdot \cos(y, n) \right) dS$$
$$- g \iiint \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] dx\,dy\,dz - \tfrac{1}{2} \iint w^2 \cdot \cos(z, n)\, dS. \tag{35}$$

(I) If S_0 is a closed cylindrical surface the generating lines of which are parallel to the z axis, and if Ω, Ω' — two functions which within S_0 satisfy the equations $L(\Omega) = 0$, $L(\Omega') = 0$ — (1) vanish at all points

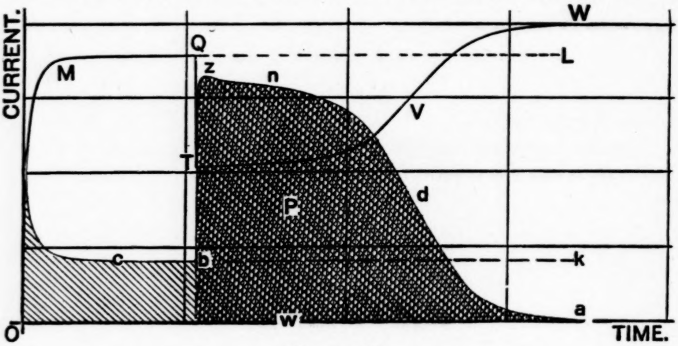

FIGURE 56.

of S_0 and at all points within S_0 for which z is positively infinite, and (2) have the given constant value Ω_0 at all points in the xy plane within S_0; then if we apply (35) to the difference between Ω and Ω', using as a field of volume integration the space inside S_0 on the positive side of the xy plane (Figure 57), we shall learn that in this space Ω and Ω' must be identically equal. The value of Ω within S_0 is in no way affected by conditions which a physical extension of the function might be required to satisfy outside S_0.

(II) If S_0 is a closed cylindrical surface, the generating lines of which are parallel to the z axis, if W is a function which within S_0 satisfies the equation $L(W) = 0$, and if

(1) W and $\partial W / \partial z$ vanish at all points within and on S_0 for which z is positively infinite,

(2) W has a given constant value (W_0) at all points on the xy plane within S_0.

(3) W on S_0 is a function (W_s) of z only, such that if n indicates the direction of the external normal to S_0

$$W_s + k\int\left(\frac{\partial W}{\partial n}\right) ds = 0, \tag{36}$$

where k is a given positive constant, and the line integral is to be taken around the perimeter of a right section of S_0 made by the plane $z = z$; and, hence, if

(4) $\iint\left(\frac{\partial W}{\partial z}\right) dS$, taken over so much of the xy plane as lies within S_0, is given, then W is uniquely determined.

If we assume that two different functions (W, W') may satisfy all these conditions, and denote their difference by u,

$L(u) = 0$, within S_0,

u and $\partial u/\partial z$ vanish at all points within S_0, for which z is positively infinite,

u vanishes at all points on the xy plane within S_0,

u on S_0 satisfies the equation

$$u_s + k\int\left(\frac{\partial u}{\partial n}\right) ds = 0. \tag{37}$$

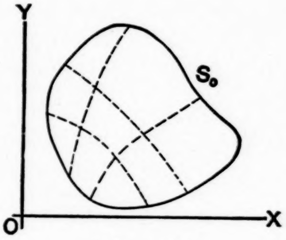

FIGURE 57.

If we use the space bounded by S_0, the xy plane, and the plane $z = \infty$, as a field of volume integration, and denote the whole boundary by S, then, since $\cos(z, n)$ vanishes on S_0, and u, $\cos(x, n)$, $\cos(y, n)$, vanish on the portions of the planes $z = 0$, $z = \infty$ used as boundaries, (35) yields the equation

$$\iiint\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy\,dz = \iint u_s \cdot \frac{\partial u}{\partial n} \cdot dS_0. \tag{38}$$

Now u has the same value at all points on the perimeter (s) of any right section of S_0, so that

$$\iint u_s \cdot \frac{\partial u}{\partial n} \cdot dS_0 = \int_0^\infty u_s \cdot dz \int \frac{\partial u}{\partial n} \cdot ds = -\frac{1}{k}\int_0^\infty u_s^2 \cdot dz, \tag{39}$$

and (38) becomes

$$\iiint\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]dx\,dy\,dz+\frac{1}{k}\int_0^\infty u_S{}^2\cdot dz=0, \qquad (40)$$

where k is intrinsically positive; but each of these last integrals has an integrand that must be either zero or positive at every point in its domain, so that u must be independent of x and y, and must vanish on S_0 at every point. It follows that u is everywhere zero and that $W = W'$.

It is evident that the condition (3) might have been stated in the form of the equation

$$W_S + k\iint\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right)dA = 0, \qquad (41)$$

where the integration is to be extended over so much of the plane $z = z$ as lies within S_0.

If the space within S_0 were cut up into portions (filaments) by the cylindrical surfaces $S_1, S_2, S_3, \cdots$, the generating lines of which were parallel to the z axis, and if within each filament $L(W)$ vanished, while, in addition to the other requirements enumerated above, W were constrained to have at every point of the surface of every filament the value (W_S), which points with the same z coordinate on the surface S_0 had, — though the normal derivative of W at the common surface of two filaments were not expected to be continuous, — we might assume as before that two different functions could satisfy all these conditions and denote their difference by u. We could then apply (35) to every filament separately (Figures 57 and 58) and obtain from each an equation of the form

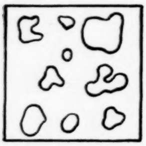

FIGURE 58.

$$\int u_S\cdot dz\iint\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)dB-\iiint\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]dz\,dB=0 \qquad (42)$$

where B denotes a cross-section of the filament. If, then, all these equations were added together, the resulting equation would be

$$\int u_S\cdot dz\iint\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)dA-\iiint\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]dz\,dA=0, \qquad (43)$$

which is (35). In this case also, therefore, W is determined.

(III) If S_0 is a closed cylindrical surface the generating lines of which are parallel to the z axis, if V is a function which within S_0 satisfies the equation $L(V) = 0$, and if

(1) V and $\partial V/\partial z$ vanish at all points within and on S_0 for which z is positively infinite,

(2) V has a given constant value (V_0) at all points on the xy plane within S_0,

(3) V on S_0 is a function (V_s) of z only, such that, if n indicates the direction of the external normal to S_0

$$V_s + l \cdot \frac{dV_s}{dz} + k \int \left(\frac{\partial V}{\partial n} \right) ds = 0,$$

or

$$V_s + l \cdot \frac{dV_s}{dz} + k \iint \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) dx\, dy = 0, \qquad (44)$$

where l and k are given positive constants, the line integral is to be taken around the perimeter (s) of a right section of S_0 made by the plane $z = z$, and the double integral over the section; then V is uniquely determined.

(IV) Let S_0 be a closed cylindrical surface which completely surrounds (Figure 58) several other mutually exclusive, closed cylindrical surfaces ($S_1, S_2, S_3, \cdots$) the generating lines of which are parallel to those of S_0 and to the z axis; and let the intersections of these surfaces with the plane $z = z$ be denoted by $s_0, s_1, s_2, s_3, \cdots$. Let the portions of the plane $z = z$ within $S_1, S_2, S_3, \cdots$, be denoted by $A_1, A_2, A_3, \cdots$, and the portion within S_0 but outside $S_1, S_2, S_3, \cdots$, be denoted by A_0. Let $\tau_0, \tau_1, \tau_2, \tau_3, \cdots$, represent the volumes of the prisms (bounded by the planes $z = 0$, $z = \infty$) of which the cross-sections made by the planes $z = z$ are $A_0, A_1, A_2, A_3, \cdots$.

In the regions $\tau_0, \tau_1, \tau_2, \tau_3, \cdots$, let the scalar function U satisfy the equations

$$\begin{aligned} \frac{\partial U}{\partial z} &= g_0 \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right), \\ \frac{\partial U}{\partial z} &= g_1 \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right), \\ \frac{\partial U}{\partial z} &= g_2 \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right), \\ &\cdots\cdots\cdots\cdots \end{aligned} \qquad (45)$$

where g_0, g_1, g_2, g_3 are given positive constants, and let the value (U_s) of U on the cylindrical surfaces be a function of z only (the same for all the surfaces), such that

$$U_s + k_0 \int\int \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) dA_0 + k_1 \int\int \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) dA_1$$
$$+ k_2 \int\int \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) dA_2 + \cdot \cdot \cdot = 0, \qquad (46)$$

where k_0, k_1, k_2, k_3 are given positive constants. Then if U has the constant value U_0 at all points in so much of the xy plane as lies within S_0 and the value zero at all points on and within S_0 for which z is positively infinite, U is determined in the positive space within S_0. For if we assume that there could be two such functions and apply (35) to their difference (u) in each of the regions $\tau_0, \tau_1, \tau_2, \tau_3, \cdot \cdot \cdot$, multiply the resultant equations by $k_0, k_1, k_2, k_3, \cdot \cdot \cdot$, and add them together, it will be easy — to show in the way indicated under (II) — that u is zero everywhere inside S_0 on the positive side of the xy plane.

It is to be remembered that

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \qquad (47)$$

is an invariant of a transformation of orthogonal Cartesian co-ordinates in the xy plane.

(V) In an important special case similar to that stated in (IV), $k_1, k_2, k_3, \cdot \cdot \cdot$, are all equal, $g_1, g_2, g_3, \cdot \cdot \cdot$, are all equal, and all the n^2 areas $A_1, A_2, A_3, \cdot \cdot \cdot$, are alike in form, however they may be oriented. In the region τ_0, U is everywhere equal to U_S, which is, as before, a function of z only, and the surface condition becomes

$$U_S + l \cdot \frac{dU_S}{dz} + k \sum_m \int\int \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) dA_m, \qquad (48)$$

where l and k are given positive constants.

If in this case we find for every one (τ_m) of the regions $\tau_1, \tau_2, \tau_3, \cdot \cdot \cdot$, the function (w_m), which within (τ_m) satisfies the equation

$$\frac{\partial w_m}{\partial z} = g \left(\frac{\partial^2 w_m}{\partial x^2} + \frac{\partial^2 w_m}{\partial y^2} \right), \qquad (49)$$

and at the boundary the surface condition

$$w_s + l \cdot \frac{dw_s}{dt} + n^2 k \iint \left(\frac{\partial^2 w_m}{\partial x^2} + \frac{\partial^2 w_m}{\partial y^2} \right) dA_m = 0, \tag{50}$$

and which has the given constant value U_0 on so much of the xy plane as lies within S_0 and the value zero when z is infinite, and if we assign to the function without S_m where it is not defined, the value zero, then, apart from differences of orientation, all these functions will be alike. If after this we define a function within S_0 by assigning to it within every one of the regions $\tau_1, \tau_2, \tau_3, \cdots$, the same value as the w function belonging to this region, and give to it in τ_0 the common value w_s, the function thus determined will be the unique function U described above.

If after a steady current of intensity E/w has been running for some time in the coil of the solenoid under consideration, so that the magnetic field within the core (which in this case shall be built up, in the manner shown in Figure 59, of filaments of square cross-sections) has everywhere the given constant value H_0, the coil circuit be very suddenly broken, the value of H falls instantly, not only at the outer surface of the prism, but also at the surface of every filament, to zero. Inside every filament

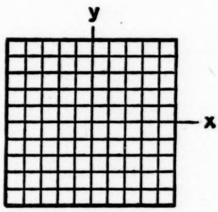

FIGURE 59.

$$\frac{\partial H}{\partial t} = \frac{\rho}{4\pi\mu} \left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} \right). \tag{51}$$

When $t = 0$, $H = H_0$ everywhere within the iron, and when t is infinite, the field intensity is everywhere zero. According to (I), therefore, we may consider every filament by itself.

If we seek a solution of the equation (51) which shall be of the form $X \cdot Y \cdot T$, where X involves x alone, Y involves y alone, and T is a function of t alone, we shall obtain the expressions

$$X = A_1 \cdot \cos \alpha x + A_2 \cdot \sin \alpha x, \quad Y = B_1 \cdot \cos \beta y + B_2 \cdot \sin \beta y, \quad T = e^{-\lambda^2 t}, \tag{52}$$

where

$$\lambda^2 = \frac{\rho(\alpha^2 + \beta^2)}{4\pi\mu}. \tag{53}$$

If we use as normal function the product

$$A_{mn} \cdot e^{-\lambda^2 t} \cdot \sin \frac{m\pi x}{c} \cdot \sin \frac{n\pi y}{c}, \tag{54}$$

where $\lambda^2 = \pi\rho(m^2 + n^2)\ (4\,\mu c^2)$ and m and n are positive integers, and write

$$H = \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} A_{mn} \cdot e^{-\lambda^2 t} \cdot \sin \frac{m\pi x}{c} \cdot \sin \frac{n\pi y}{c}, \tag{55}$$

this expression will satisfy all conditions if A_{mn} be so taken that when $t = 0$, the second number of the equation shall be equal to H_0 for all values of x and y within the filament. We have, therefore, the equation[14]

$$A_{mn} = \frac{4\,H_0}{c^2} \int_0^c d\chi \int_0^c \sin \frac{m\pi\chi}{c} \cdot \sin \frac{n\pi\psi}{c}\, d\psi \tag{56}$$

and

$$A_{mn} = \frac{16}{\pi^2} \frac{H_0}{mn},$$

when m and n are both odd;

$$A_{mn} = 0,$$

when either m or n is even, so that

$$H = \frac{16\,H_0}{\pi^2} \sum_{j=0}^{j=\infty} \sum_{k=0}^{k=\infty} \frac{e^{-\lambda^2 t}}{(2\,j+1)(2\,k+1)} \cdot \sin \frac{(2\,k+1)\pi x}{c} \cdot \sin \frac{(2\,j+1)\pi y}{c} \tag{57}$$

$$\lambda^2 = \frac{\pi\rho}{4\,\mu c^2}[(2\,k+1)^2 + (2\,j+1)^2]. \tag{58}$$

From (58) it appears that the whole flux of magnetic induction through the core at the time t is

$$\phi = \frac{64 \cdot \mu \cdot H_0 \cdot c^2}{\pi^4} \sum_{j=1}^{j=\infty} \sum_{k=1}^{k=\infty} \frac{e^{-\lambda^2 t}}{(2\,j+1)^2(2\,k+1)^2}, \tag{59}$$

or, if

$$g = \pi\rho/4\,\mu c^2,$$

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[14] Byerly, Treatise on Fourier's Series, etc., § 71. Riemann-Weber, Die partiellen Differential-gleichungen der mathematischen Physik, Bd. II, § 99.

$$\phi = \frac{64 \cdot \mu \cdot H_0 \cdot c^2}{\pi^4} \sum_{j=1}^{j=\infty} \frac{e^{-g(2j+1)^2 t}}{(2j+1)^2} \sum_{k=1}^{k=\infty} \frac{e^{-g(2k+1)^2 t}}{(2k+1)^2} \tag{60}$$

In these equations absolute electromagnetic units are to be used, and for good soft iron we may assume that $\pi\rho/4$ is very approximately equal to 8000. It is evident that for different values of c when μ is given, $e^{-\lambda^2 t}$ will have the same numerical value for values of t proportional to c^2; for instance, if $c = 20$, $t = 10$, $e^{-\lambda^2 t}$ will have the same value as it would if c were 1 and t, 1/40. If c is fixed, $e^{-\lambda^2 t}$ will have the same value for values of t proportional to μ.

It is possible to show that if $c = 1$ and $\mu = 200$, — to take a special case, — the series

$$S \equiv \sum_{k=0}^{k=\infty} \frac{e^{-g(2k+1)^2 t}}{(2k+1)^2} \tag{61}$$

has at different times the approximate values given in the following table:

TABLE V.

t.	S.	t.	S.
0	1.2337	0.01000	0.6734
0.00025	1.1450	0.02000	0.4494
0.00050	1.1084	0.02500	0.3679
0.00100	1.0565	0.05000	0.1353
0.00200	0.9830	0.07500	0.04979
0.00250	0.9534	0.10000	0.01832
0.00500	0.8374	0.20000	0.00034

From the numbers in this table it is easy to compute, for cores of square cross-section, the fractional part of the original induction flux through the core which remains after the circuit of the exciting coil has been broken for a given time. For a solid core, the area of the square section of which is 100 square centimeters, the results are given in the next table, when μ is 200.

If the core were built up compactly of varnished square rods of one square centimeter in cross-section, the times in the table should be

divided by 100, and if the core were made up of 10,000 slender filaments, the flux would sensibly disappear during the first thousandth of a second. It is easy to get similar results for any other value of μ.

TABLE VI.

Time in Seconds after the Breaking of the Circuit.	Fractional Part of Original Flux still remaining.	Time in Seconds after the Breaking of the Circuit.	Fractional Part of Original Flux still remaining.
0.000	1.000	1.000	0.298
0.025	0.861	2.000	0.133
0.050	0.807	2.500	0.089
0.100	0.733	5.000	0.012
0.200	0.635	7.500	0.0016
0.250	0.597	10.000	0.0002
0.500	0.461		

If the cross-section of the core were a circle of radius a, and if, after a uniform magnetic field of strength H_0 had been established in the core the exciting circuit were suddenly broken, the intensity of the field at any time, at any point distant r centimeters from the axis would be given by the expression[15]

$$H = \frac{2H_0}{a} \cdot \sum_k \frac{J_0(n_k \cdot r)}{n_k \cdot J_1(n_k \cdot a)} e^{-\beta_k^2 t} \qquad (62)$$

where $\beta^2 = \rho n^2 / 4\pi\mu$ and the whole flux through the core would be

$$2\pi\mu \int_0^a Hrdr \text{ or } 4\pi\mu H_0 \sum_k \frac{e^{-\beta_k^2 t}}{n_k^2}. \qquad (63)$$

In these equations $n_k a$ is the kth root in order of magnitude of the Bessel's Equation

$$J_0(na) = 0. \qquad (64)$$

[15] Heaviside, Electrical Papers, **1**, xxviii. Peirce, These Proceedings, **41**, 1906. Byerly, Treatise on Fourier's Series, etc., p. 229.

The first ten roots are as follows:

TABLE VII.

k.	na.	k.	na.
1	2.404826	6	18.071064
2	5.520078	7	21.211637
3	8.653728	8	24.352472
4	11.791534	9	27.493479
5	14.930918	10	30.634606

From these numbers the β's can be found, and then from (63) the flux in the core after any interval. When the time is short, the series converges very slowly, and the computation is long and troublesome, but for relatively large values of t the work is not difficult.

The next table shows the fractional part (Ω) of the original flux remaining in a core, the cross-section of which is a circle of 20 centimeters diameter, and in which μ is 200; 1 second, 4 seconds, and 8 seconds after the breaking of the exciting circuit: the corresponding fraction for a core of square cross-section (20 cms. $\times$ 20 cms.) is given for comparison. The actual value of the original flux is of course a little larger in the second case because the area of the cross-section is greater.

TABLE VIII.

t.	Ω for the Round Core.	Ω for the Square Core.
1	0.588	0.597
4	0.270	0.298
8	0.106	0.133

After 16 seconds Ω for the round core would be 0.016. In the case of a round core of exactly the same cross-section area as the square solid core, and the same original flux, the fractional part remaining after one second would be 0.630.

If the square core of the solenoid — the area of the cross-section of which is A square centimeters — be made of a bundle of infinitely long,

straight iron wires, placed close together (Figure 60), and if, after a steady current of intensity E/w has been running for some time through the solenoid, so that there is a magnetic field of uniform intensity $H_0 = 4\pi NE/w$ in the core, the applied electromotive force be suddenly shunted out of the solenoid circuit, the current (C) in the coil will gradually die out. At any instant the field, in so much of the space A as is occupied by air, is $4\pi NC$, for eddy currents in the wires act like solenoid sheets and do not affect the field without the wires. Within each wire there are eddy currents, of course, and at every point in the wire, at every instant, the field intensity, H, must satisfy the equation

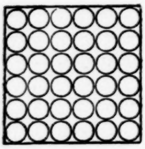

FIGURE 60.

$$\frac{\partial H}{\partial t} = \frac{\rho}{4\pi\mu}\left[\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right]. \tag{65}$$

The induction flux through the turns of the solenoid per centimeter of its length shall be p, so that

$$E - \frac{dp}{dt} = wC, \quad \text{or, in this case,} \quad \frac{dp}{dt} = -wC.$$

If there are n^2 wires in the core and the area of the cross-section of each of them is B,

$$p = 4\pi N^2 C(A - n^2B) + \mu N \iint H \cdot dx\,dy \tag{66}$$

where the double integral is to be extended over the cross-sections of all the wires; hence

$$wC + (A - n^2B)\,4\pi N^2 \cdot \frac{dC}{dt} + \mu N \iint \frac{\partial H}{\partial t} \cdot dx\,dy = 0; \tag{67}$$

and if the wires fill the square space as full as possible,

$$A - n^2B = 0.2146\,A, \text{ nearly.}$$

If H_s represents the strength of the magnetic field in the air space within the solenoid,

$$H_s + \frac{4\pi N^2}{w}(A - n^2B)\frac{dH_s}{dt} + \frac{4\pi\mu N^2}{w}\iint \frac{\partial H}{\partial t} \cdot dx\,dy = 0. \tag{68}$$

The function H thus defined falls under theorem (V) above, and it is evident that we ought to seek, for a single wire, a function ϖ which within the wire shall satisfy (65), at the surface shall fulfil the condition

$$\varpi_S + \frac{4\pi N^2}{w}(A - n^2B)\frac{d\varpi_S}{dt} + \frac{4\pi\mu n^2 N^2}{w}\iint \frac{\partial\varpi}{dt}\cdot dx\,dy = 0, \qquad (69)$$

and which when $t = 0$ shall have the value H_0 and when t is infinite, the value zero. When we have to deal with a single wire of radius $b (= a/n)$ alone, it is obviously convenient to use polar co-ordinates with origin at the point where the axis of the wire cuts the xy plane, and if we do this (65) and (67) take the forms

$$\frac{\partial\varpi}{\partial t} = \frac{\rho}{4\pi\mu r}\cdot\frac{\partial}{\partial r}\left[r\cdot\frac{\partial\varpi}{\partial r}\right], \qquad (70)$$

$$\varpi_S + \frac{4\pi N^2}{w}(A - n^2B)\frac{d\varpi_S}{dt} + \frac{2\pi n^2 N^2 \rho b}{w}\left(\frac{\partial\varpi}{\partial r}\right)_{r=b} = 0, \qquad (71)$$

or

$$\varpi_S + l\cdot\frac{d\varpi_S}{dt} + kn^2 b\left(\frac{\partial\varpi}{\partial r}\right)_{r=b} = 0, \qquad (72)$$

where l, k, n, and b are given, positive constants.

If we attempt to find a solution of (70) in the form of the product of a function of t, and a function of r, we arrive, of course, at the normal form

$$e^{-\beta^2 t}[L\cdot J_0(mr) + M\cdot K_0(mr)], \qquad (73)$$

but Bessel's Functions of the second kind will not be needed here, and we may write, $M = 0$,

$$\varpi = \sum_m L_m\cdot e^{-\beta^2 t}\cdot J_0(mr), \qquad (74)$$

where either m or β may be assumed at pleasure and the other computed from the equation

$$m^2\rho = 4\pi\mu\beta^2. \qquad (75)$$

If for m in the equation (74) we use the successive roots of the transcendental equation

$$J_0(mb) = \frac{kn^2\cdot mb}{1 - l\beta^2}\cdot J_1(mb) \qquad (76)$$

the series will satisfy (70) and (72), and if the coefficients can be so chosen as to make

$$\sum_{0}^{\infty} L_m \cdot J_0(mr) = H_0 \tag{77}$$

equation (74) will give the function sought.

Although the development (77) is not one of those for which the coefficients can be found by the usual devices, it is easy to solve the problem, for such cases as are of practical interest, to any desirable approximation.

We shall find it instructive, however, to inquire first what the solution would be if the second term of (72) were lacking, for, in view of the fact that the permeability of the iron is relatively large compared with that of the air, it seems likely that in some instances, where the series is very convergent, this modified problem and the real one will have nearly equal numerical answers.

We have, then, so to choose L_m, β, and m, subject to (75) that the value of the series (77) shall be H_0 when $t = 0$, for all values of r up to b; and that at every instant

$$\varpi_S + \frac{2\pi n^2 N^2 \rho l}{w}\left(\frac{\partial \varpi}{\partial r}\right)_{r=b} = 0. \tag{78}$$

It is necessary, therefore, that m shall be a root of the transcendental equation

$$J_0(mb) = \frac{2\pi N^2 n^2 \rho}{w} \cdot mb \cdot J_1(mb), \tag{79}$$

which may be written in other forms by virtue of the relations

$$\frac{dJ_0(x)}{dx} = -J_1(x), \qquad \int_0^x x \cdot J_0(x)\,dx = x \cdot J_1(x). \tag{80}$$

It will be convenient to illustrate the effect of making b small (and therefore n large) while a is kept constant, by a numerical example. Let us assume that the cross-section of the solenoid is a square of 10 centimeters side-length, so that $a = 5$; let the solenoid have 10 turns of insulated wire per centimeter of its length, and let the resistance of these 10 turns be $\frac{1}{16}$th of an ohm, so that in absolute units $w = 10^9/16$. If, then, we take the specific resistance of the core to be $(10^6/32\pi)$

absohms at the room temperature (Fleming and Dewar), $2\pi N^2\rho/w$ will be equal to $\frac{1}{10}$, and the equation for m takes the form

$$J_0(mb) = \frac{n^2}{10}(mb) \cdot J_1(mb) = \frac{mb}{\lambda} \cdot J_1(mb). \qquad (81)$$

But[16]

$$1 = \sum \frac{2\lambda \cdot J_0(mr)}{(\lambda^2 + m^2b^2) J_0(mb)}, \qquad (82)$$

and hence

$$\varpi = 2\lambda H_0 \sum_m \frac{e^{-\beta^2 t} \cdot J_0(mr)}{(\lambda^2 + m^2b^2) J_0(mb)}. \qquad (83)$$

The whole flux of magnetic induction through the iron of the core is then μn^2 times the integral of ϖ taken over the circle of radius b in which ϖ is defined; that is

$$\phi = 4\pi\mu\lambda H_0 n^2 b \sum \frac{e^{-\beta^2 t} \cdot J_1(mb)}{m(\lambda^2 + m^2b^2) J_0(mb)}, \qquad (84)$$

or

$$\phi = 4\pi\mu\lambda^2 H_0 n^2 \sum \frac{e^{-\beta^2 t}}{m^2(\lambda^2 + m^2b^2)}. \qquad (85)$$

Since $\lambda = 10/n^2$, the coefficient of the series may be written $400\,\pi\mu H_0/n^2$, and we may assume that $\mu = 100$.

The time rate of change of the total induction flux through the turns of the solenoid, per centimeter of its length, is

$$\frac{9950 \cdot 10^4 \cdot H_0}{n^2} \sum \frac{e^{-\beta^2 t}}{\lambda^2 + m^2b^2}. \qquad (86)$$

If the square core is built up of 100 circular rods, each 1 centimeter in diameter, $n^2 = 100$, $\lambda = 1/10$, and the m's are defined by the equation

$$J_0(mb) = 10\,mb \cdot J_1(mb) \qquad (87)$$

in which $b = 1/2$.

It is not difficult to show by trial and error from Meissel's tables[17] that the first five roots of this equation have values approximately equal to those given in the following table:

[16] Byerly, Treatise on Fourier's Series, etc., p. 229.

[17] Meissel, Tafel der Bessels'schen Functionen, Berliner Abhandlungen, 1888; Gray and Mathews, Treatise on Bessel's Functions, pp. 247–266; Peirce and Willson, Bulletin of the American Mathematical Society, 1897.

TABLE IX.

$m_1b = 0.44168$	$\log \beta_1^2 = 0.79077$	$m_1^2 = 0.78032$
$m_2b = 3.858$	$\log \beta_2^2 = 2.6733$	$m_2^2 = 59.527$
$m_3b = 7.030$	$\log \beta_3^2 = 3.1946$	$m_3^2 = 197.672$
$m_4b = 10.183$	$\log \beta_4^2 = 3.5164$	$m_4^2 = 414.798$
$m_5b = 13.331$	$\log \beta_5^2 = 3.7504$	$m_5^2 = 710.884$

A mere inspection of these values shows that the value of ϕ can be computed with an accuracy much more than sufficient for any practical purpose from the first two terms of the series (85), if t is as great as $\frac{1}{100}$th of a second, and from the first term alone if t is as great as $\frac{1}{40}$th of a second. Let ϕ_0 represent the first term of (85), then

$$\phi_0 = \frac{400\,\pi H_0\, e^{-6.1768t}}{(0.78032)(0.20508)},$$

but

$$\frac{400}{(0.78032)(0.20508)} = 2499.55, \tag{88}$$

which differs from 2500 by about $\frac{1}{50}$th of one per cent only.

If there were no eddy currents in the iron, the total induction flux through the rods which make up the core would be

$$\phi' = \pi\mu a^2 H'_s, \tag{89}$$

and if C' were the strength of the current in the exciting coil at the time t, we should have

$$\pi\mu a^2 N \cdot \frac{dH'_s}{dt} = -\,w \cdot C' = \frac{-\,w \cdot H'_s}{4\,\pi N} \tag{90}$$

and

$$H'_s = H_0\, e^{-ht}, \tag{91}$$

where

$$h = w/4\,\pi^2 N^2 a^2 \mu = 6.332573 +$$

and

$$\phi' = \pi\mu a^2 H_0 e^{-ht}. \tag{92}$$

In the case under consideration we should have very nearly

$$\phi' = 2500\,\pi H_0\, e^{-6.332573t} \tag{93}$$

$$4\,\pi N C' = H'_s = H_0\, e^{-6.332573t}. \tag{94}$$

When there are eddy currents the value of H_S is given with sufficient accuracy by the first term of (83) very soon after the electromotive force has been shunted out of the circuit, that is by the equation,

$$H_S = \frac{2000}{2051} \cdot H_0 \, e^{-6.1768t} \tag{95}$$

and the ratio of ϕ to $\pi b^2 n^2 \mu H_S$ is practically equal to the constant 2051/2000, for it is easy to find a very convergent geometrical series every term of which is greater than the corresponding term of the series which begins with the second term of (85), and the sum of this geometrical series is extremely small except for very small values of t.

According to this analysis, the current in the solenoid will have fallen in the first second to the fraction 0.002025 or to the fraction 0.001777 of its original value according as there are or are not eddy currents in the iron.

If the ten centimeter square iron core of the solenoid were built up of straight rods only one millimeter in diameter, we should have $b = 1/20$, $n = 100$, and $\lambda = 1/1000$; the m's would need to be roots of the equation

$$J_0(mb) = 1000\, mb \cdot J_1(mb). \tag{96}$$

By using differences of the third order it is possible to show from Meissel's table that the first root is approximately equal to 0.044715 + and the second to 3.83. For the first, then, $\lambda^2 + m^2 b^2 = 0.002000$, and $\beta^2 = 6.33077$. For the second root, $\beta^2 = 46500$, and the second terms of the series (83) and (85) become negligible almost immediately after the electromotive force has been removed from the circuit.

In this case

$$\phi_0 = 2500\, \pi H_0 \cdot e^{-6.33077t} \tag{97}$$

very nearly; and

$$\frac{C}{4\pi N} = H_S = H_0 \cdot e^{-6.33077t}, \tag{98}$$

so that the disturbing effects of the eddy currents are comparatively slight. At the end of one second, the current will have fallen to the fraction 0.001777 of its original value or to the fraction 0.001781, according as eddy currents were absent or existent. These differ by only about one two hundred and fifty thousandth part of the original current strength. We may note in passing that a very approximate value (correct to four significant figures) of the first root of the equation might be found by equating to unity the coefficient of the first term of the series (83).

If the core of the solenoid were made of wire one tenth of a millimeter in diameter, such as is now in common use in coils intended for loading long telephone circuits, we should have $b = 1/200$, $n = 1000$, $\lambda = 1/100000$, and m would need to satisfy the equation

$$J_0(mb) = 100000\, mb \cdot J_1(mb). \tag{99}$$

It is easy to see that the first root of this has a value very nearly equal to 0.0044721, and that the effects of eddy currents would be quite inappreciable.

Having considered somewhat at length — on the supposition that the induction flux in the air spaces of the core might be neglected — the manner in which a current in the solenoid would decay if the electromotive force were suddenly removed from the circuit without changing the resistance, we may now return to the more general case to which the equations (74) and (76) belong, and remark that in the ideal case where eddy currents are supposed to be absent (68) takes the form

$$H'_S + \frac{4\,\pi N^2}{w}(21.46)\frac{dH'_S}{dt} + \frac{4\,\pi\mu N^2 n^2 \pi b^2}{w} \cdot \frac{dH_S}{dt} = 0, \tag{100}$$

whence $$H'_S = H_0 \cdot e^{-6.31567t}. \tag{101}$$

It is clear at the outset that the larger roots, at least, of the two equations (76) and (79) will be very different, since the second member of (76) soon has a negative coefficient. If then the coefficients of the series (77) could be found, the series (74) and (83) would not resemble each other in appearance for large values of b and small values of the time. If, however, b is fairly small, as it usually is in practice, we may dismiss all thought of the infinite series, since it is easy to choose the coefficients of two or three terms of the form (73) so that the initial condition shall be satisfied very approximately. In many cases one term suffices.

Let us consider first the case — already treated in another way — of a square core of 100 square centimeters cross-section, built up of long straight wires 1 millimeter in diameter; so that $b = 1/20$, $n = 100$, $l\beta^2 = 1.36620\, m^2b^2$, $kn^2 = 1000$, and the equation for mb has the form

$$J_0(x) = \frac{1000\,x}{1 - 1.36620\,x^2} J_1(x). \tag{102}$$

It is possible to show by a rather long application of the method of trial and error, using third differences in Meissel's table, that the value of the first root is 0.044654+ and this corresponds to $m = 0.89308$, $\beta^2 = 6.31351$, $J_0(mb) = 0.9994891+$.

If, then, we consider the single term

$$Q = H_0 e^{-6.31351t} \cdot J_0 (0.89308\, r), \tag{103}$$

Q will satisfy (70) and will vanish when t is infinite. When t is zero, Q will be equal to H_0 for $r = 0$, and will differ from H_0 by about one twentieth of one per cent when $r = b$. The second root of (102) is roughly equal to 3.8 and the corresponding value of β^2 is about 45,000, so that the exponential factor would soon be very small. An inspection of the graph of $J_0(x)$ shows that if we were to use several terms of the form $L \cdot e^{-\beta^2 t} \cdot J_0(mr)$, we could easily form a function which should differ very little from H_0 for any value of r up to b, when t was zero; but it is clear that after the lapse of about 1/5000th of a second, all the terms beyond the first would be negligible, and there is no practical advantage in using more than one term.

We may assume then that the value of H in any one of the iron rods is given fairly accurately, except at the very beginning, by (103). Since $4\pi NC = H_S$ the current in the solenoid falls in the first second to 0.001808 of its original value, or to 0.001812 times that value according as eddy currents are absent or present. These fractions differ from each other by about one two hundred and fifty thousandth part of the original current strength. Another close approximation to the value of H may be made by dividing (103) by $J_0(mb)$ and another by multiplying the second member of (103) by

$$\frac{1 + J_0(mb)}{2\, J_0(mb)}. \tag{104}$$

These changes would not affect the relative rate of decay of the current.

The nearness of the approximation to the value of the field attainable by a single term is evidently much increased as the diameter of the iron wire of which the core is built up is decreased. If as before $a = 5$, but if $b = 1/200$, $n = 1000$, the value of the first root of the equation for mb will be 0.00446616, nearly, and the value of $J_0(mr)$ will not change by so much as 1/100000th part of itself as r changes from 0 to b. A single term, therefore, will represent H with great accuracy. In this case the effect of eddy currents is wholly inappre-

ciable. Of course this statement does not apply to the case of an alternate current of very great frequency.

In the problem just considered the electromotive force was suddenly shunted out of the solenoid circuit after a steady current had been established in it, and, on the assumption that the permeability of the iron was fixed, the value of the magnetic field within the core was determined as a function $[H_0 f(t, r)]$ of the time and the space coordinates. The function f satisfies (65) and (68), vanishes when t is infinite, and is initially equal to unity. If the solenoid circuit containing an applied electromotive force E be suddenly closed at the time $t = 0$, and if the ultimate value $(4\pi NE/w)$ of the magnetic field in the core be denoted by H_∞, the value of the field at any time will be given by the equation

$$H = H_\infty [1 - f(t,r)]. \tag{105}$$

The function defined by this equation vanishes, when $t = 0$, for all values of r, and when t is infinite is equal to H_∞. It satisfies at all times the equation (65) and the surface equation

$$H_s + \frac{4\pi N^2}{w}(A - n^2 B)\frac{dH_s}{dt} + \frac{4\pi\mu N^2}{w}\int\int\frac{\partial H}{\partial t}dx\,dy = \frac{4\pi N}{w}\cdot E, \tag{106}$$

and such a function is evidently unique.

Although in practice the permeability is not fixed, the analysis of this section enables us to shut in between narrow limits the effects of eddy currents in many cases, and to assert, when this is the truth, that in a given instance the effects of such currents will be negligible, if the pieces of which the core is built are properly varnished.

It is sometimes possible to get interesting information about the magnetic properties of the core of a transformer which has several coils, and about the excellence of the insulation of the sheets of which it is made, by observing the sudden changes in the currents in the coils when the inductances of the system are impulsively changed, or by studying the rate of propagation of the induction flux into the core, but these subjects must be left for the next instalment of this paper.

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